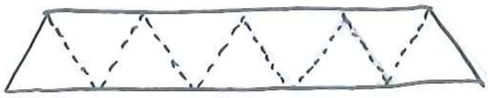
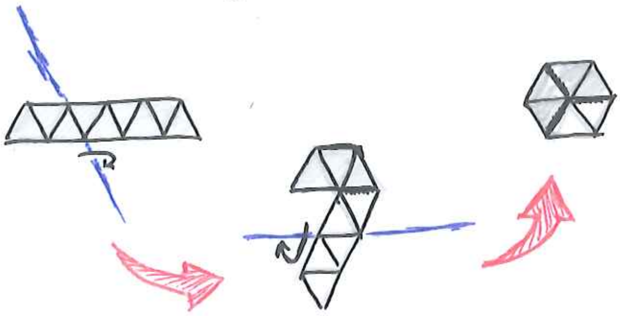
How to build a hexaflexagon

Step 1 Cut out a strip of nine equilateral triangles.

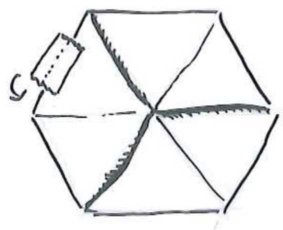


Step 2 Fold the dotted edges sharply in both directions. They should bend easily.

Step 3 Fold the strip twice as indicated, tucking the final triangle up.

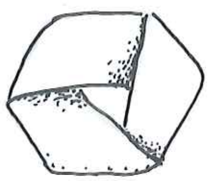


Step 4 Apply cello tape along north-~~east~~west edge.



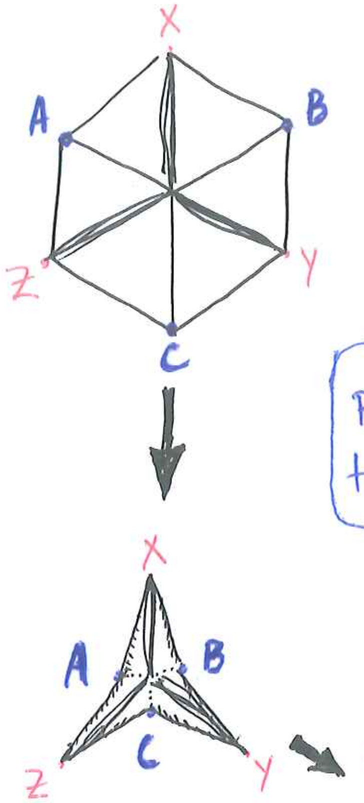
NOTE: These instructions will lead to a "left handed" flexagon.

"left-handed"

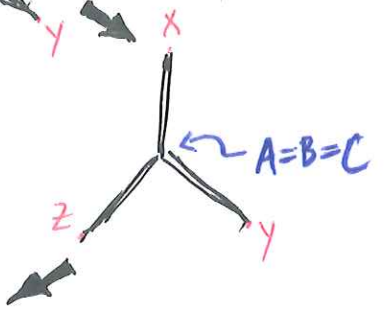


"right-handed"

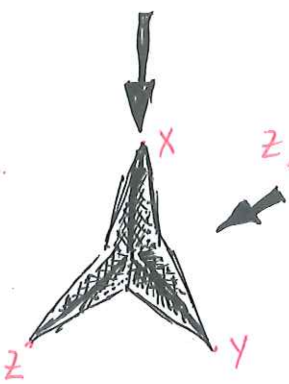
How to "flex" a hexaflexagon



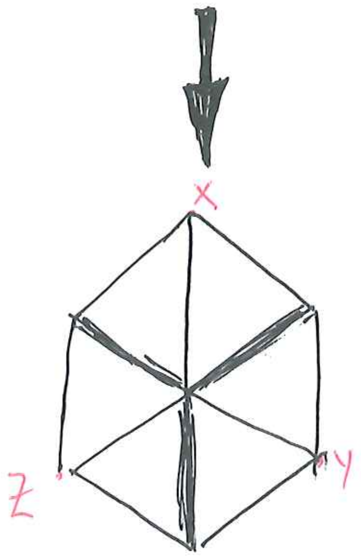
Pinch points A, B, and C together to a point behind the hexagon (the double edges will become mountain folds).

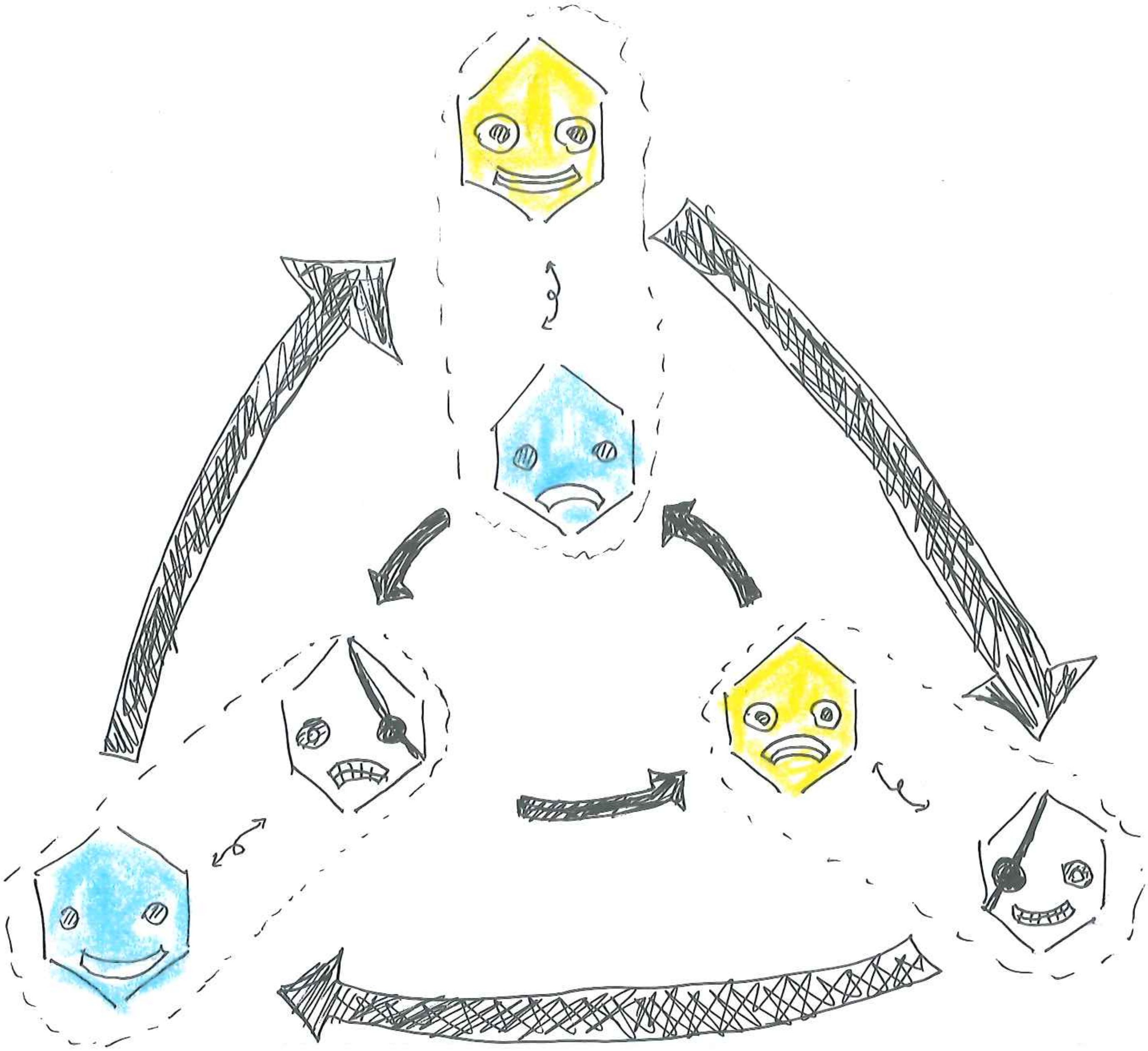


The top should then open out from the center.



If you haven't rotated the vertices in the process, you should see the double edges are in new locations.





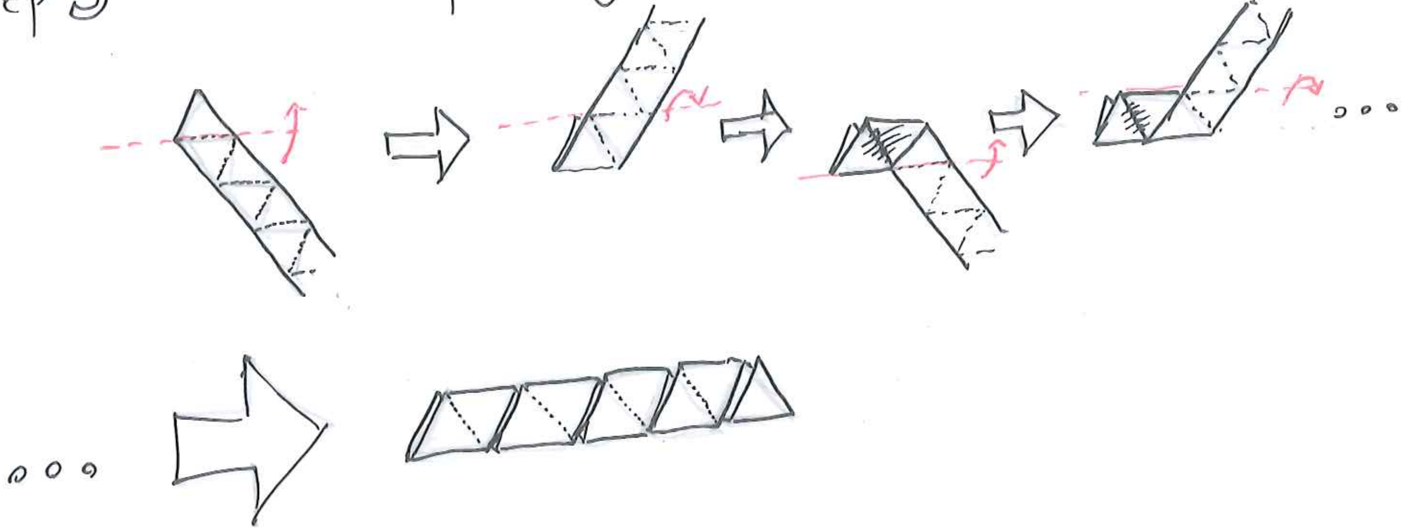
States of the tri-hexaflexagon

How to build a hexahexa flexagon

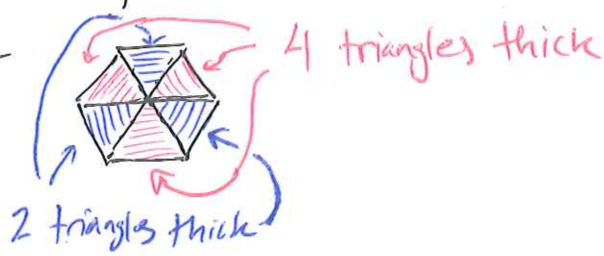
Step 1 Cut out an 18 triangle strip, or tape two 9 triangle strips together.

Step 2 Fold all edges well in both directions.

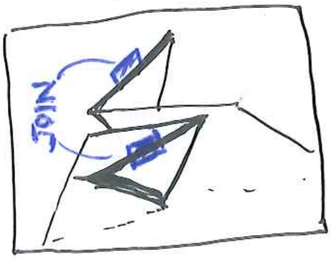
Step 3 Wind it up to get a doubled 9 triangle strip:



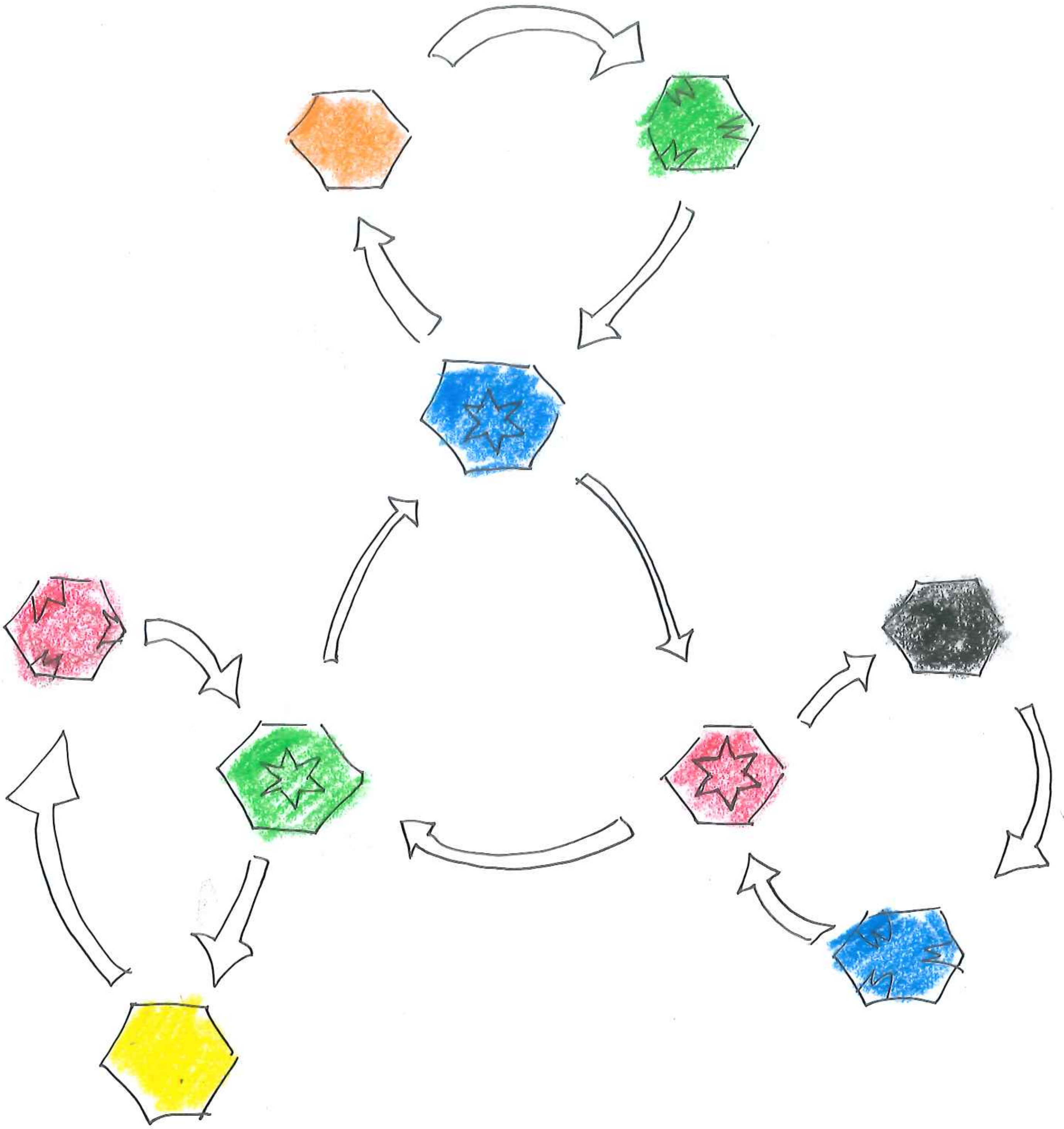
Step 4 Now repeat the construction of the trihexaflexagon to get



Step 5 Apply cello tape to the North West edge, but note



you should just tape the outer two edges together.



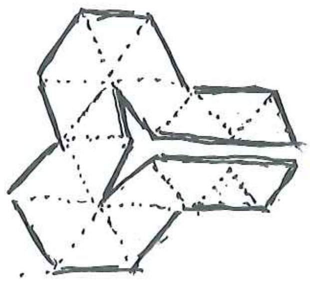
State diagram of hexa-hexa flexagon
(ignoring the underside)

Are there other hexaflexagons?

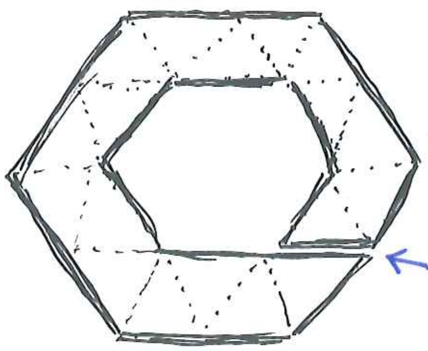
We will call a flexagon "order-n" if there are n disjoint faces.

There are actually <u>three</u> distinct <u>order-6</u> hexaflexagons:

① The "straight-strip" 

② 

EXERCISE: Determine the state diagrams.

③ 

NOTE: These gaps were introduced in the drawing for clarity. Actually all triangles are equilateral.

The Theory of Flexigation

In 1940, Feynman and Tukey worked out a full theory of flexigation, in which they showed that

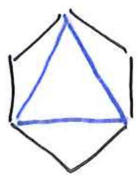
$$\boxed{\text{the number of order-}n\text{ hexaflexagons}}$$

$$\|$$

$$\boxed{\text{the number of triangulations of a regular } n\text{-gon}}$$

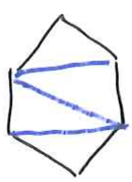
For example, there are three triangulations of a regular 6-gon (i.e. hexagon):

(1)



(2)

(3)



Quiz!

① How many times must we flex the trihexaflexagon to get back to our exact starting state, if we don't allow rotations of the hexagon?

② What is the fastest way to visit all faces of the hexa-hexaflexagon?

③ How many distinct order-7 hexaflexagons are there?

Some questions / problems:

1. Construct an order-4 hexaflexagon.

2. Prove that there exists an order $3 \cdot 2^n$ hexaflexagon for any $n \geq 0$, made from a straight-strip.

3. Prove that there exists an order $3n$ hexaflexagon made from a straight-strip for any $n \geq 0$.

4. Prove that there exists at least one order n hexaflexagon for any $n \geq 3$ (not necessarily straight-strip).

5. How many order-n-hexaflexagons are there (with distinct folding patterns)?