Pascal's Triangle Evened Out

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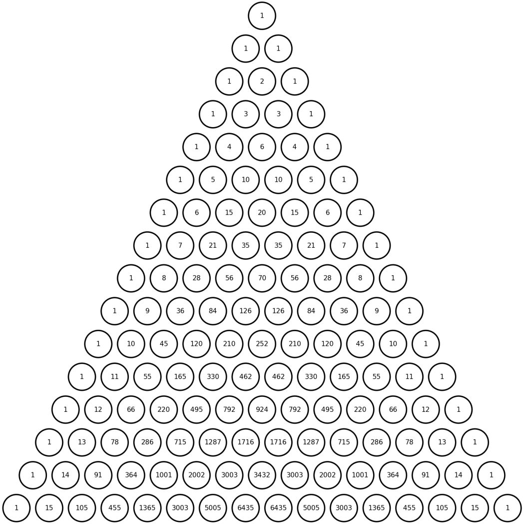
CMI

July 18, 2016

What is Pascal's Triangle?

It is easiest to see in a picture:



Who created it?

One of the most remarkable aspects of the triangle is how often it has appeared in different cultures and contexts and time periods.

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Pascal

Pascal collected many results about the triangle and applied them to probability theory.

Omar Khayyam

Omar Khayyam extended the work of Al-Karaji, and introduced binomial coefficients.

Yang Hui

Yang Hui extended the work of Jia Xian.

Pingala

Pingala studied the triangle, which he called *The Staircase of Mount Meru*, in connection with binomial coefficients and combinatorics.

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7

Warm-Up Quiz

Let's review some properties of Pascal's Triangle. None of these will be used after the quiz–so don't worry if you haven't seen these properties before!

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- Can you find the triangular numbers?
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- What is the sum of entries in row n?
- Can you *prove* that the sum of entries in row n is 2^n?
- What is the sum of *squares* of entries in row n?
- Can you find the triangular numbers?
- Can you find the Fibonacci numbers?

Something to Think About

Question

How many odd entries are in row n of Pascal's triangle?

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Figure: Drawing by Karen Haydock

It may be useful to color the odd entries. Use the sample triangle and the crayons provided.

How to Draw Pacal's Triangle mod 2?

- You *could* start with Pascal's Triangle over the integers, and then simply color the even and odd integers accordingly.

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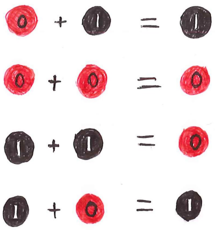
- But you could also start with an un-numbered triangle, and use the simple (mod 2) addition rules to produce each row from the previous one:



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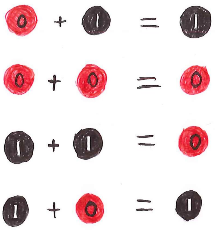


- Are there other ways to construct Pascal's Triangle mod 2?

Back to the question: Let's make some guesses!

Question

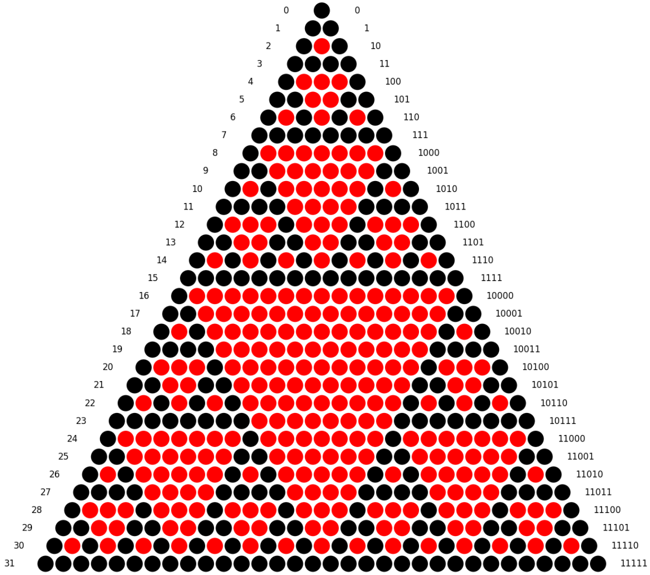
How many odd entries are in row n of Pascal's triangle?

Don't worry about making mistakes. Sometimes wrong guesses will lead to the right answer.

Figure: Drawing by Karen Haydock

Let's write out your conjectures on the board.

Notice the binary expansion of row n...



You guessed it!

Theorem

There are exactly $2^{\#(n)_2}$ odd entries in the n^{th} row of Pascal's triangle, where $\#(n)_2$ denotes the number of 1's in the binary expansion of n.

But how can we prove it?

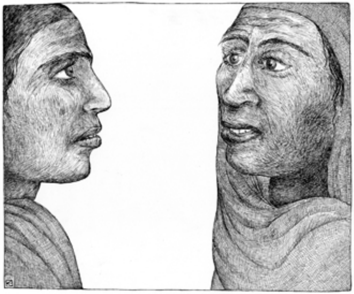
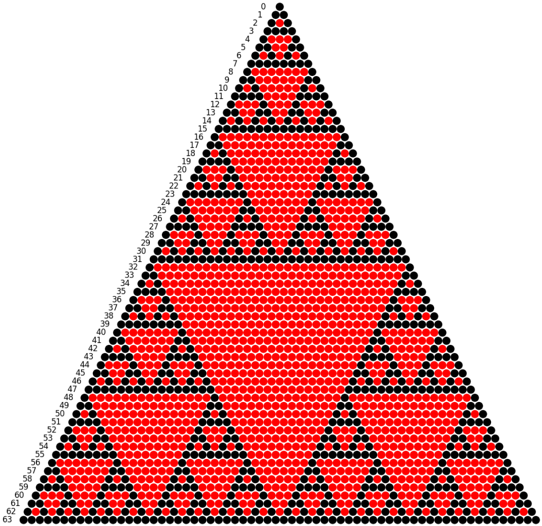


Figure: Drawing by Karen Haydock

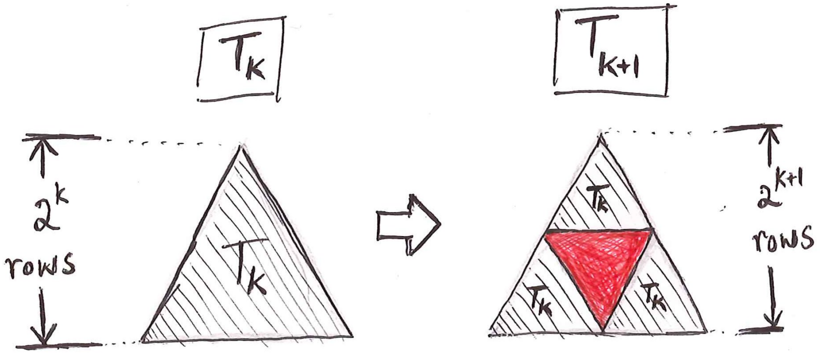
Let's write some of your ideas on the board.

How to Construct Pacal's Triangle mod 2?

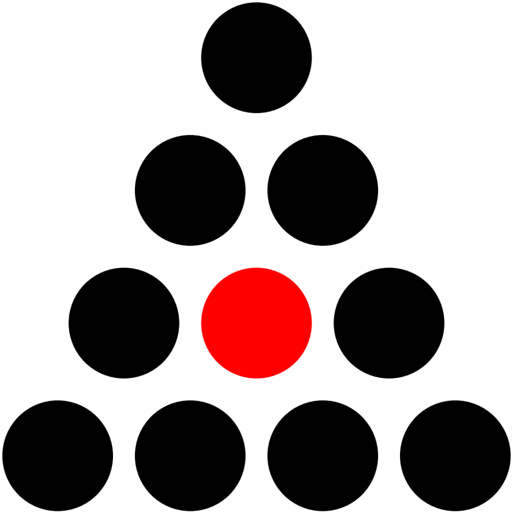
Here are the first 64 rows. Is there a faster way to construct many rows?



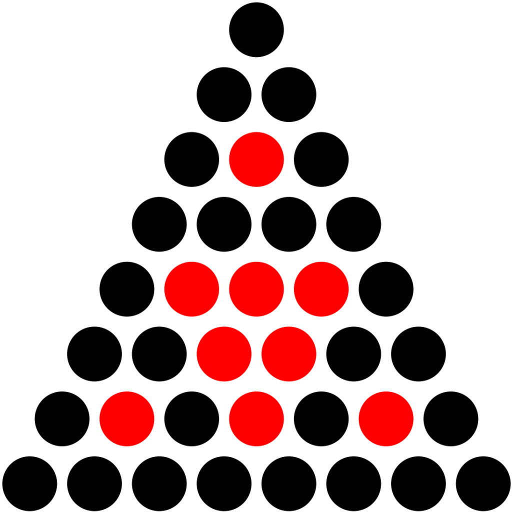
Growth diagram (mod 2)



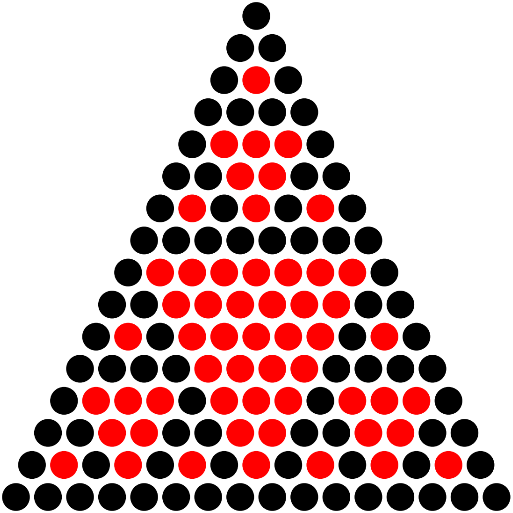
T_2



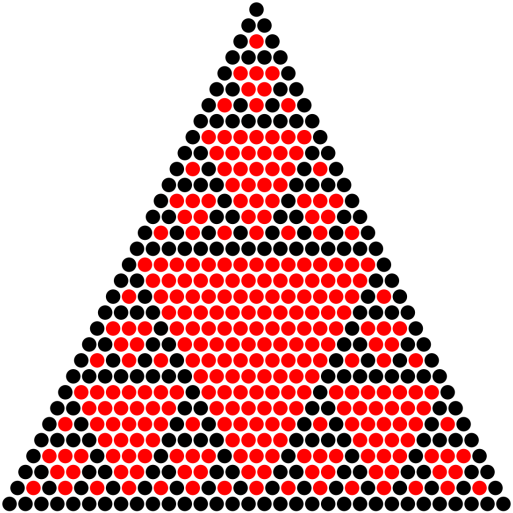
T_3



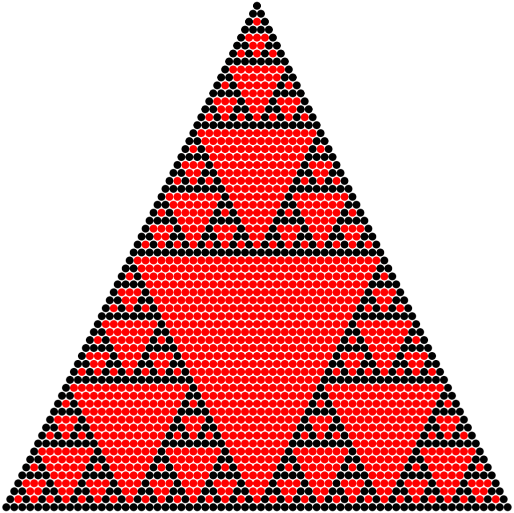
T_4



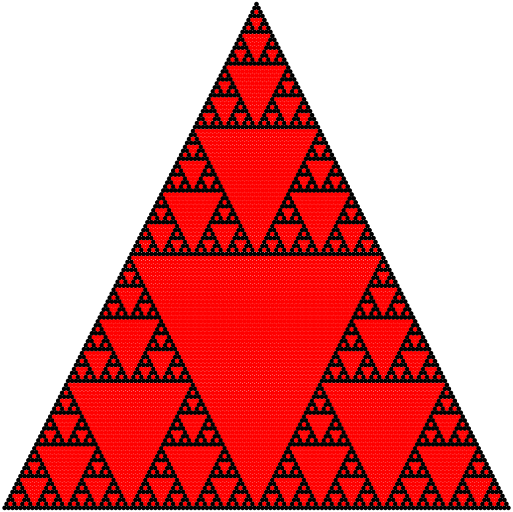
T_5



T_6



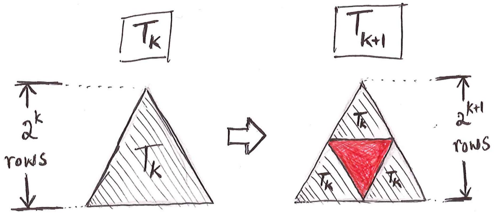
T_7



The Proof

Question

How can we use the growth diagram to help us prove the result?



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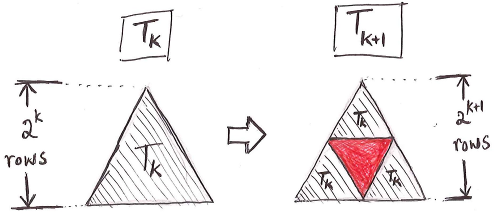
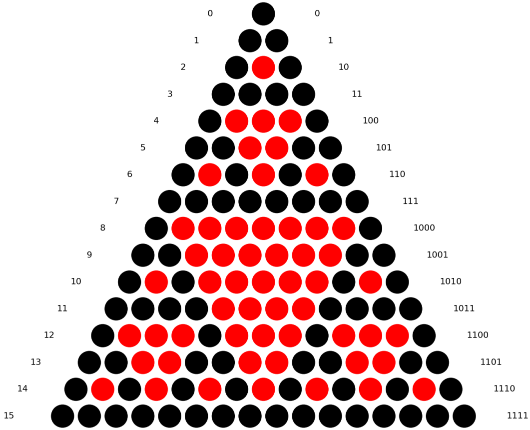


Answer

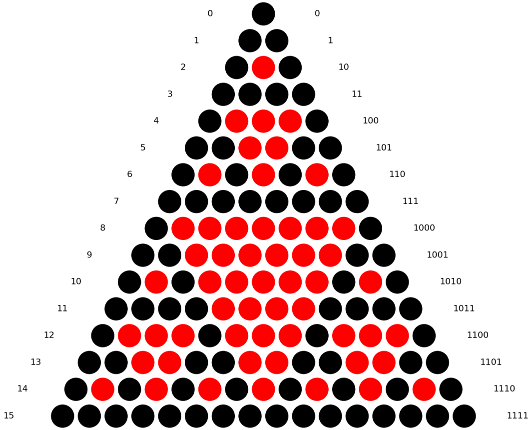
We can prove the result by induction on k: in other words, we can show that if we believe the result for the triangle T_k, then we should believe it for T_{k+1} as well.

The Proof



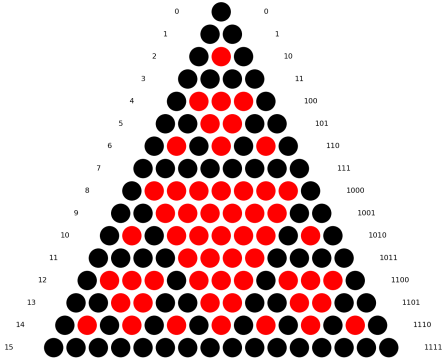
For example, suppose the result holds for all rows R_m such that $m < 2^3 = 8$ (i.e. all rows in the triangle T_3).

The Proof



For example, suppose the result holds for all rows R_m such that $m < 2^3 = 8$ (i.e. all rows in the triangle T_3).
Let R_n be a row of T_4 that is *not* contained in T_3 (i.e. $8 \leq n \leq 15$).

The Proof



- Notice that R_{n-2^3} is contained in T_3.
- Also, by our construction of T_4, there are twice as many odd entries in row R_n as in row R_{n-2^3}.
- Finally, the binary expansion of $n - 2^3$ is obtained by removing the initial 1 from the binary expansion of n.

The Proof

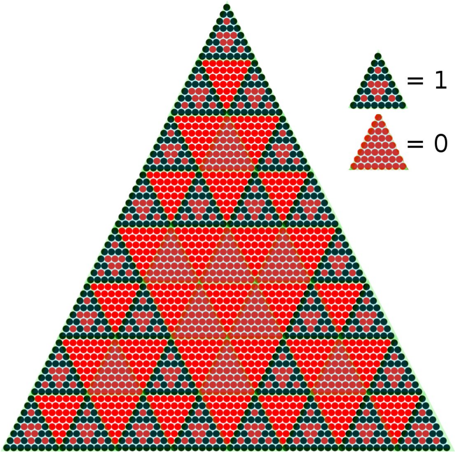
Putting all this together, we have:

Proof.

$\#$ of odd entries in row R_n
$= 2 * (\#$ of odd entries in row $R_{n-2^k})$ by construction of T_{k+1}
$= 2 * 2^{\#(n-2^k)_2}$ by inductive hypothesis
$= 2^{\#(n-2^k)_2+1}$
$= 2^{\#(n)_2}$ by property of binary expansion. □

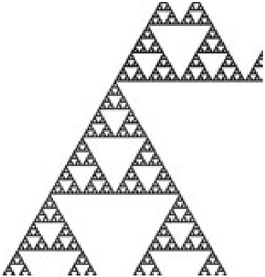
Self-similarity

Pascal's triangle (mod 2) exhibits a property known as *self-similarity*. One aspect is the following: for any positive k you can let the equilateral triangles of height 2^k be the *basic units* instead of 1 and 0. You will still get the same configuration!



Self-similarity

There's a name for this configuration: *Sierpinski's gasket*.



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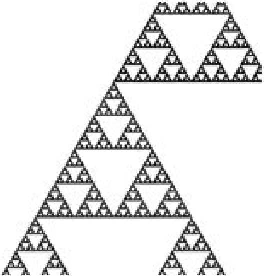

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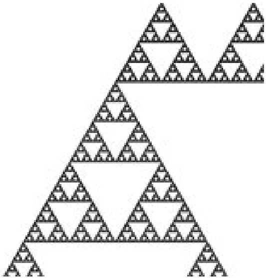

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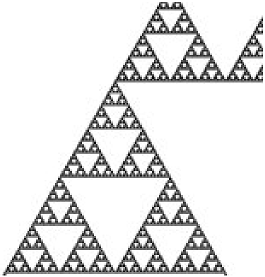

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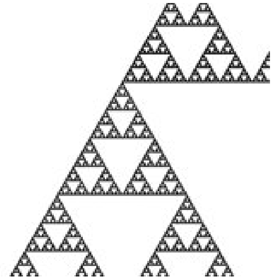

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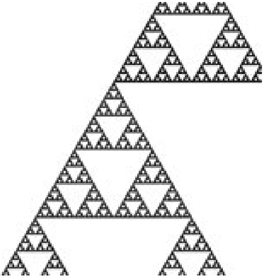

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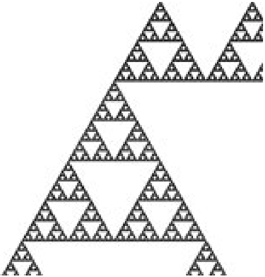

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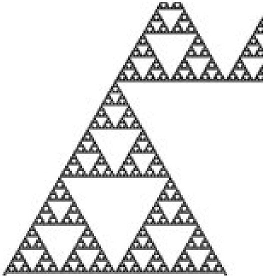

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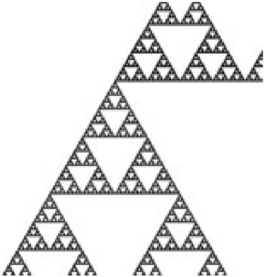

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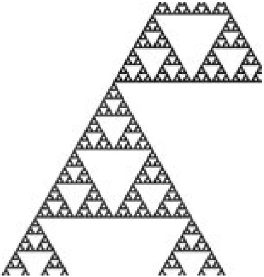

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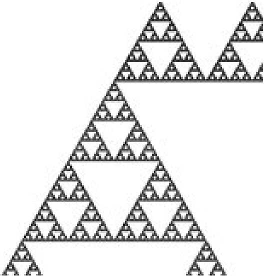

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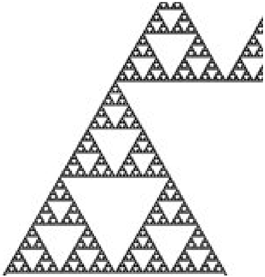

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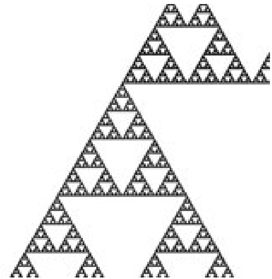

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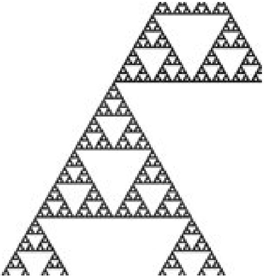

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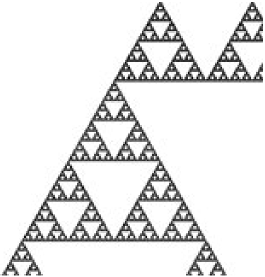

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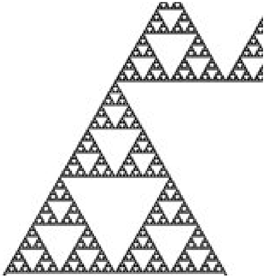


Self-similarity also occurs in nature

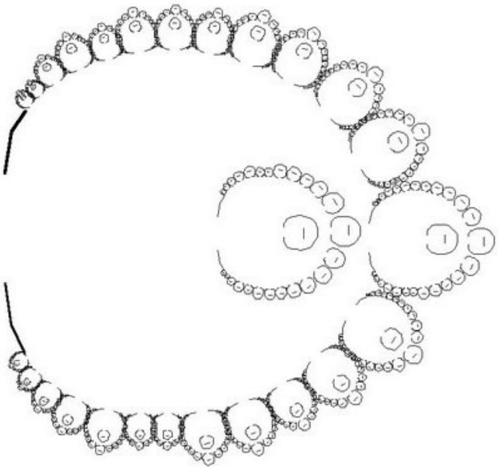
For example on the patterns of certain seashells:

Self-similarity also occurs in nature

Or in the shape of ferns:

Self-similarity in culture

Almost all human cultures make use of self-similar patterns. For example some traditional West African villages follow a fractal layout:



Returning to Pascal's Triangle...(mod 3)!

Question

Can you figure out the (mod 3) growth pattern?

- As a first step, color the first 9 rows (mod 3)...

Returning to Pascal's Triangle...(mod 3)!

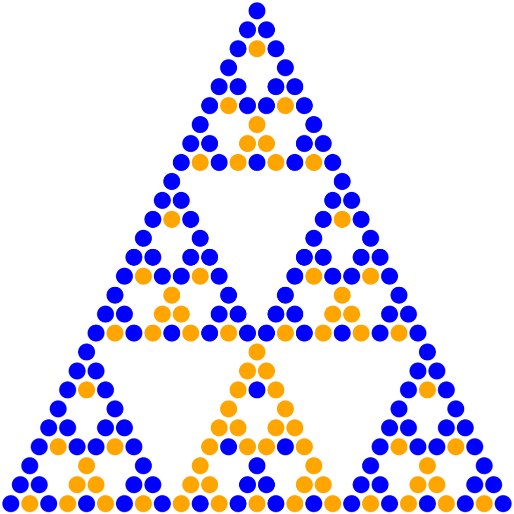
Question

Can you figure out the (mod 3) growth pattern?

- As a first step, color the first 9 rows (mod 3)...
- Perhaps you can guess the pattern already?

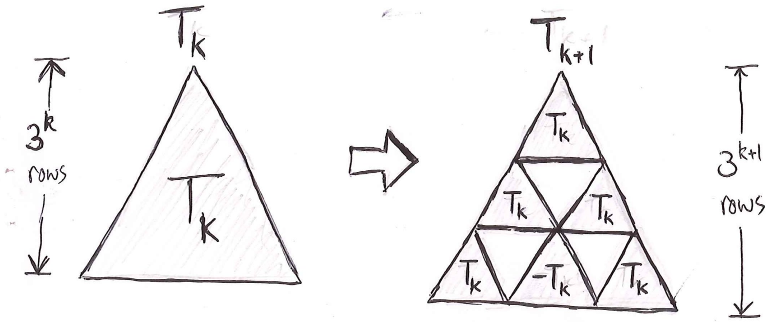
Returning to Pascal's Triangle...(mod 3)!

Here are the first 27 rows to help you:



Returning to Pascal's Triangle...(mod 3)!

The growth diagram turns out to be:



What about odd entries (mod 4)?

Question

Can you figure out the (mod 4) growth pattern?

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Can you figure out the (mod 4) growth pattern?

This is more difficult...but it's helpful to *just focus on the odd entries*.

What about odd entries (mod 4)?

Question

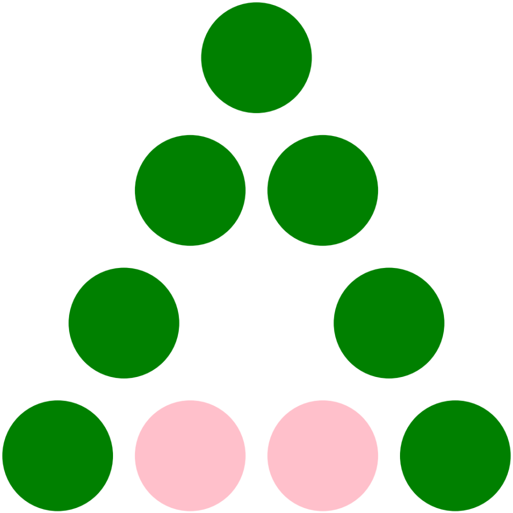
Can you figure out the (mod 4) growth pattern?

This is more difficult...but it's helpful to *just focus on the odd entries*. Another question to think about:

Question

Of the odd entries in a given row, how many are $\equiv 1 (\text{mod } 4)$ *and how many are* $\equiv 3 (\text{mod } 4)$*?*

What about odd entries (mod 4)?



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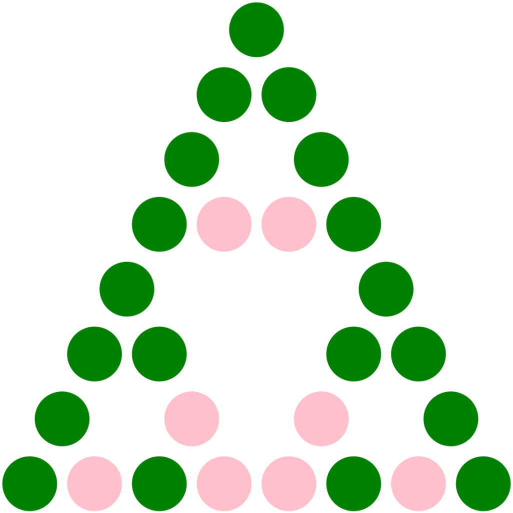

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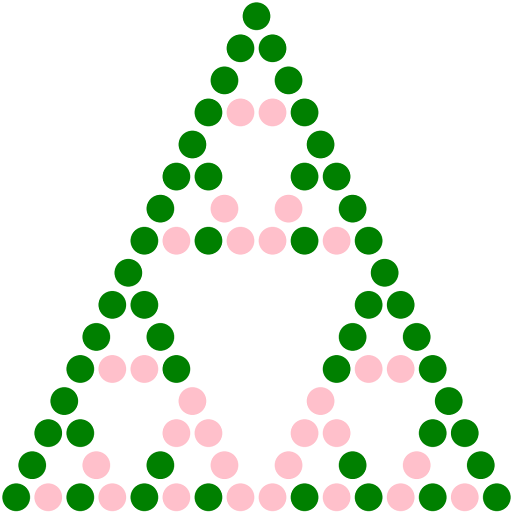

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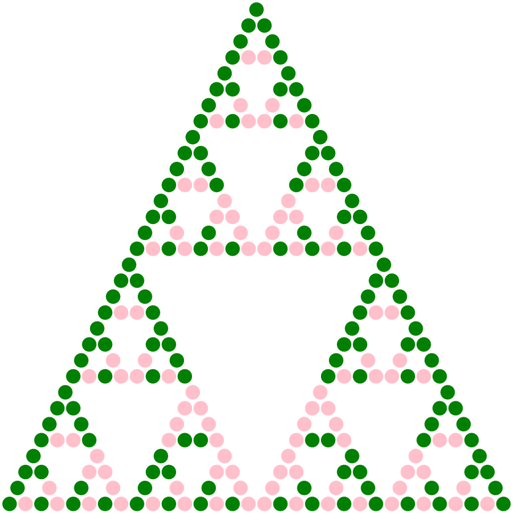


What about odd entries (mod 4)?

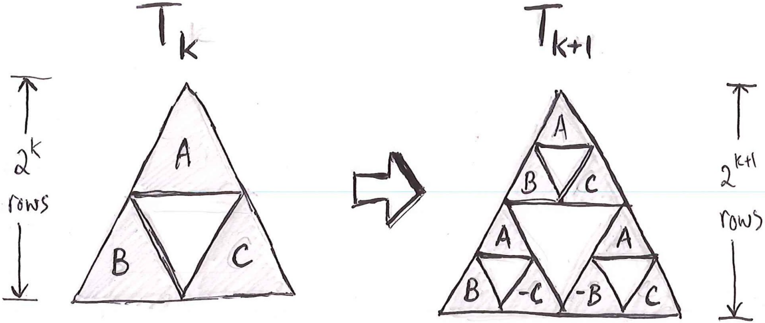
We even have the following theorem, which you can try to prove on your own:

Theorem

The number of entries $\equiv 1 (mod\ 4)$ equals the number of entries $\equiv 3 (mod\ 4)$ in row n if and only if there are two consecutive 1's in the binary expansion of n; otherwise there are no entries $\equiv 3$ in row n.

What about odd entries (mod 4)?

It turns out the growth pattern is the following for the odd entries. See if you can use it to prove the theorem!



What about (mod 8)?

There is a similar theorem for Pascal's triangle (mod 8) but remarkably it fails for (mod 16).
See the paper *Zaphod Beeblebrox's brain and the fifty-ninth row of Pascal's triangle* by Andrew Granville for the details. Most of this talk was based on that paper as well!