Geometry of vision

Vijay Ravikumar

CMI

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Here's an old painting

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The Tribute Money; Italy 1426 CE, Masaccio.
It's credited as one of the first painting to feature linear perspective.

This one is much newer

Untitled, France 1914 CE, Maurice Utrillo

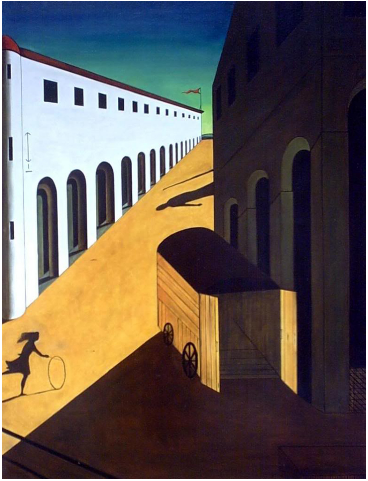
And this is somewhere in the middle

The Music Lesson, France 1665, Johannes Vermeer

Not all art uses linear perspective

My mother's vision, 1981, Joane Cardinal-Schubert

Some artists play around with it



The melancholy and mystery of a street, 1914, de Chirico

Others blatantly subvert it

Perspective Should Be Reversed, 2014, David Hockney

And for some it just doesn't seem relevant

Bird on Money, 1981, Jean-Michel Basquiat

Does this painting use perspective?

Untitled, India, early 1600's, Artist Unknown.

What about this one?

The birth of St. Edmund, England, late 1400's, Artist Unknown

What about this one?

Hint: look at the tiles

How do these tiles compare?

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Which tiles are more square?

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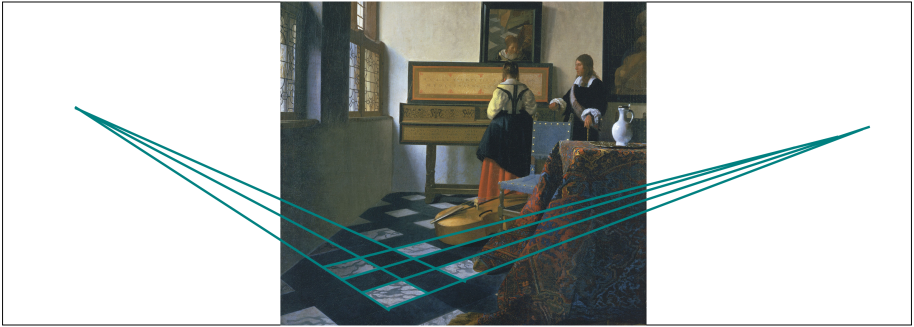
Vermeer's tiles don't have right angles and they change size. Yet they look like perfect squares! Why?

The key discovery



The images of parallel lines should converge.
When we observe it in action, our brain somehow knows it is 'correct'.

The key discovery

This might not seem surprising today due to modern photography, and with so many long straight lines to observe.
But it took a long time to discover, and had a radical effect on art!

The key discovery

In any case it's not immediately obvious why parallel lines converge?
Do you have an answer?

An experiment

Have you drawn in perspective before?

An experiment

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Let's try to do it.

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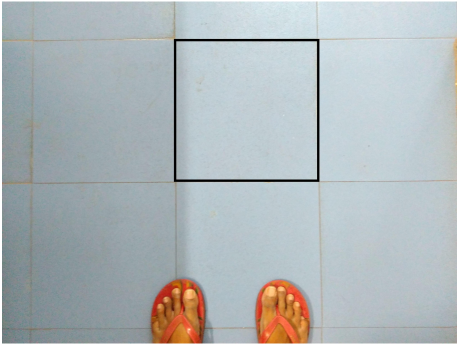

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But if you raise your head slightly, it might look a bit more interesting:

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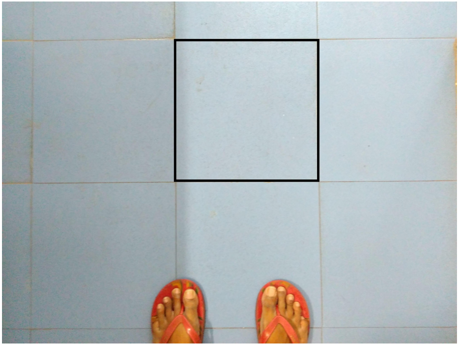
More precisely, suppose you were given the image of a single tile, and asked to extend it to a tiling of the entire floor.
Which tools would you need to draw each of the above images?

An experiment

Let's start with the first image:



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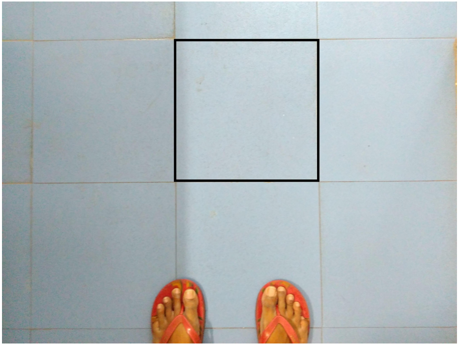

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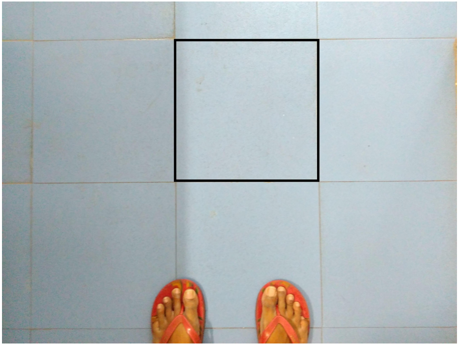

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First extend the edges of the tile.

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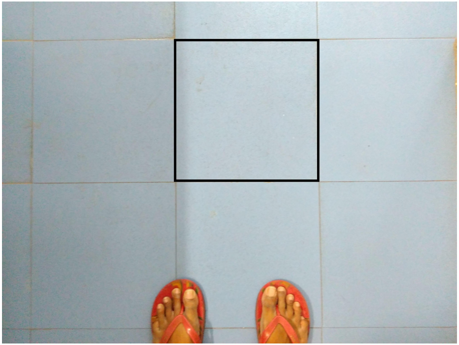

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First extend the edges of the tile.
Then evenly measure out corner points along each line.

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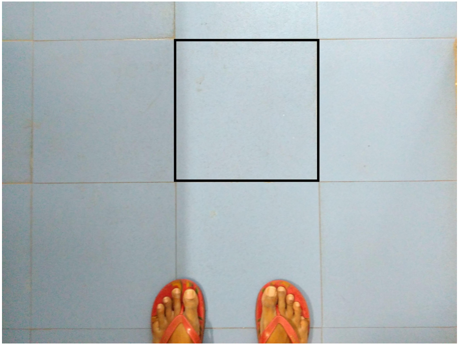

You would need a compass and straightedge.
First extend the edges of the tile.
Then evenly measure out corner points along each line.
Finally connect the corners to form the grid.

An experiment

What about the second image?

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Again you would need a compass and straightedge.

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First extend the bottom edge, and evenly measure out corner points along it.

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What about the second image?

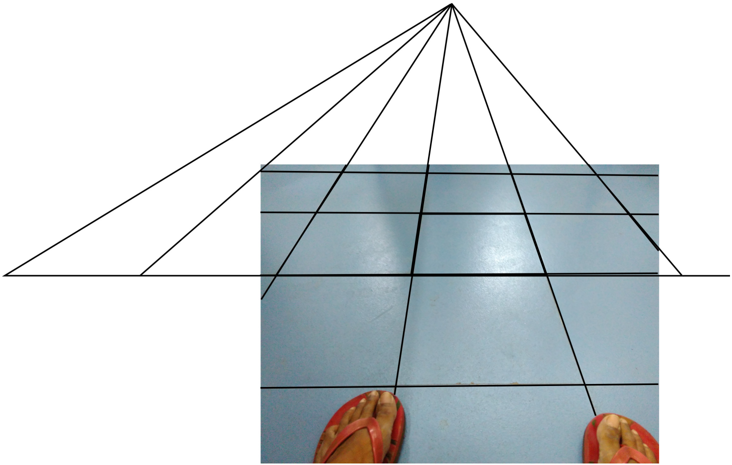
Again you would need a compass and straightedge.

First extend the bottom edge, and evenly measure out corner points along it.

Now extend left and right edges of the tile upward, until they meet.

We call this point a *vanishing point*.

An experiment



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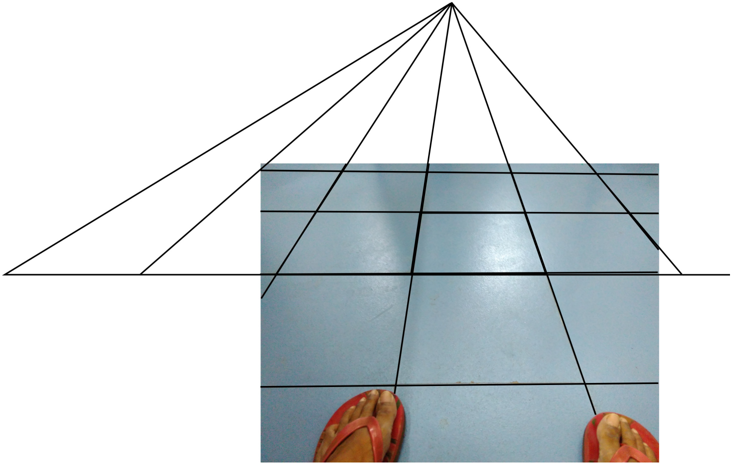
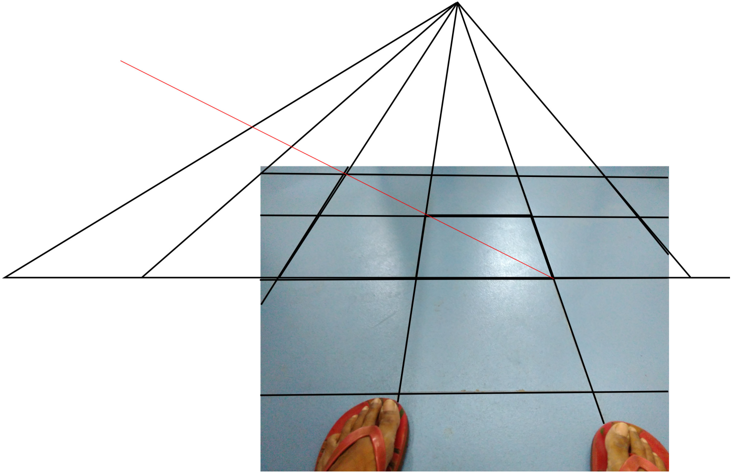


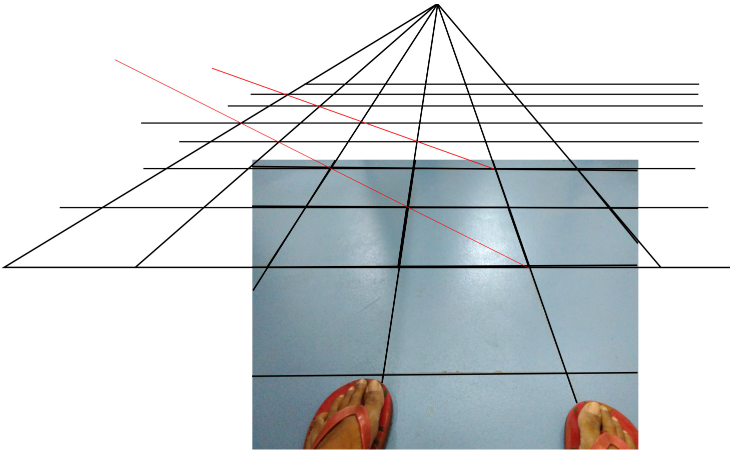
Now connect the corners to the vanishing point.
But how to complete the grid with horizontal lines?

An experiment



Simply draw a diagonal through the tiles!
It must intersect the 'vertical' lines at corners.

An experiment



Voila!

You can draw more diagonals if needed.

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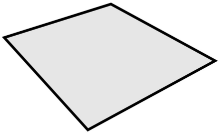
Surprisingly, you can complete the tiling with a straightedge alone. No compass needed!
Can you figure out how to do it?

The Exercise!

Complete the perspective view of the tiled floor, using only an unmarked straightedge.



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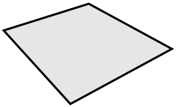

When you are finished, try drawing a tiling of your own on the back side of the sheet.

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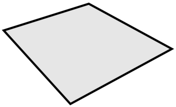
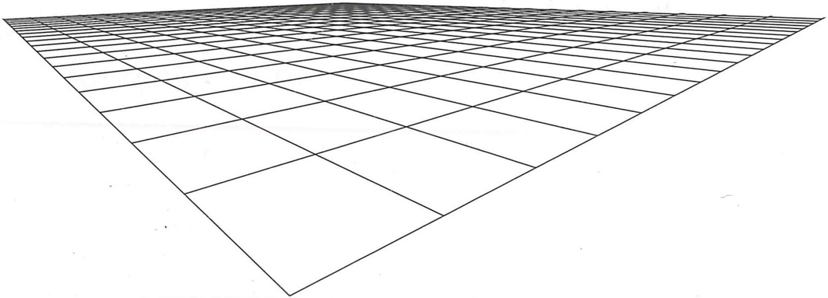


When you are finished, try drawing a tiling of your own on the back side of the sheet.
Hint: start by fixing a horizon line and two vanishing points.

Woo hoo!



We really can make a perspective view of a tiled floor, using only a straightedge!

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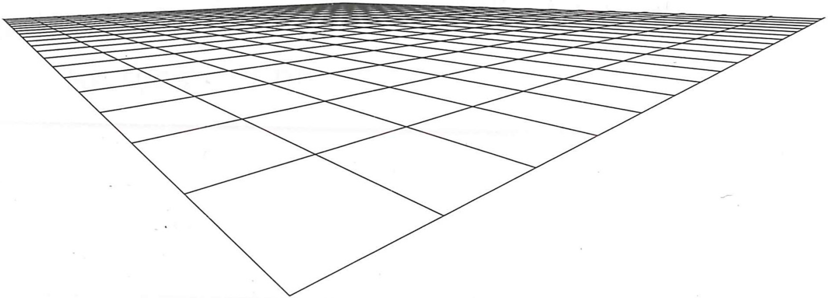

There are some natural questions to ask though:

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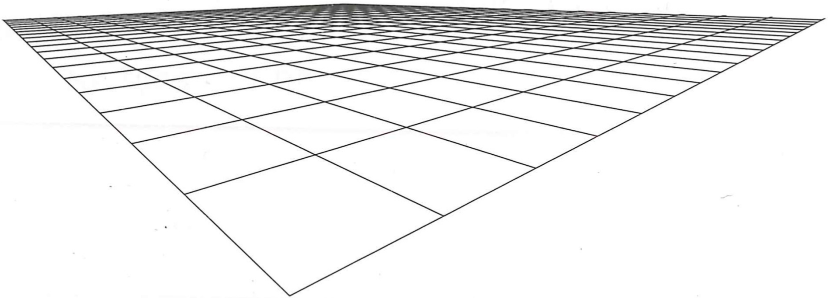

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1. Why does it look so realistic? How can we tell the tiles have the same size despite their depictions having different areas?

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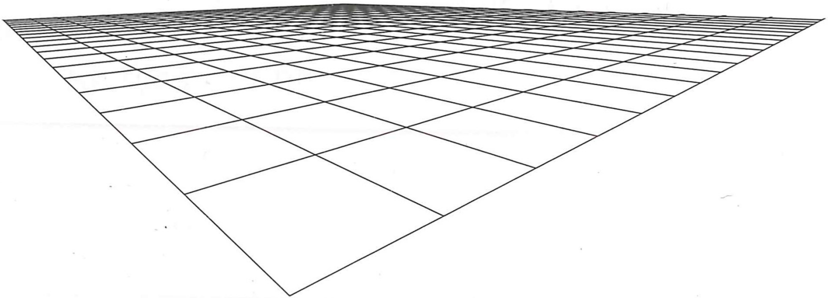

There are some natural questions to ask though:

1. Why does it look so realistic? How can we tell the tiles have the same size despite their depictions having different areas?
2. Why was this view of the tiled floor *easier* to draw than the other two?

What do these images have in common?

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Although distances and angles change, *something* must also stay the same when we change perspective.
After all, we can somehow *tell* that all these images represent the same grid.

What do these images have in common?

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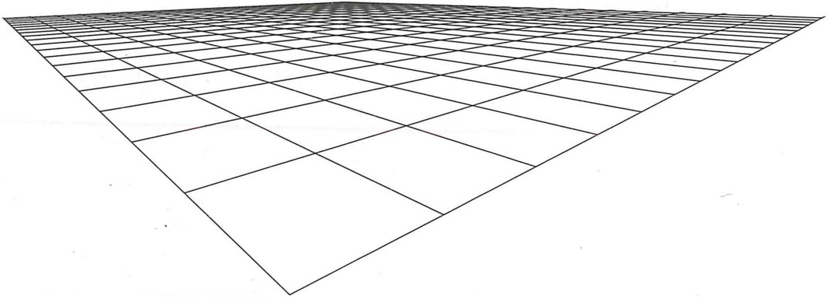
What stays the same?
Straight lines remain straight.
Intersections remain intersections.
Parallel lines remain parallel or meet at the *horizon*.

Points at infinity



Let's pretend for a moment that these vanishing points, or *points at infinity* are not just imaginary, but actually exist. And that the *horizon* is an actual line consisting of all of them.

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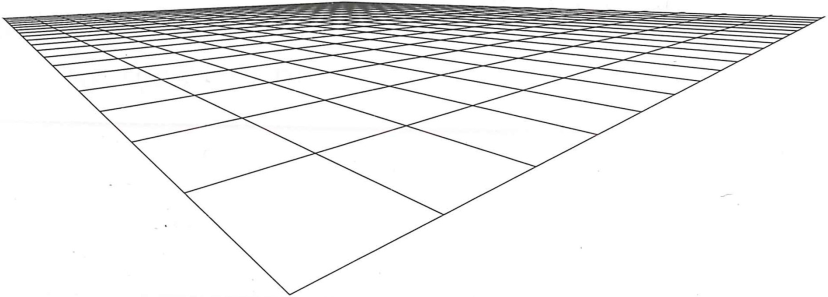

Let's pretend for a moment that these vanishing points, or *points at infinity* are not just imaginary, but actually exist. And that the *horizon* is an actual line consisting of all of them.
It seems quite elegant: any two points determine a line. And any two lines intersect in a single point.

Well, almost...

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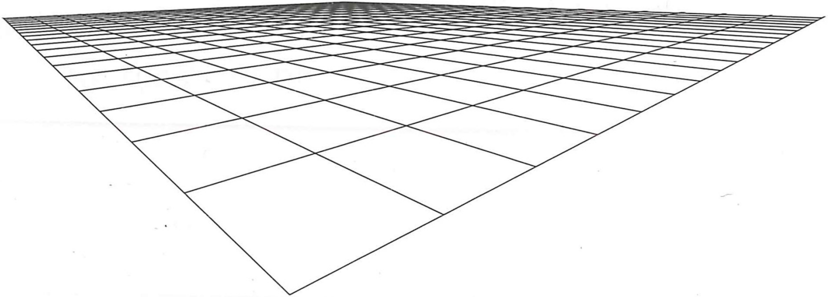
We may have to turn a little, to see the point at infinity where these formerly horizontal lines meet.

What if?



Can there exist a geometry in which any two lines intersect in a point? And any two points determine a line? Is so perhaps we can model our vision on it.

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1. Any two *projective points* are contained in a unique *projective line*.

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3. There exist four *projective points*, no three of which are in a *projective line*.

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For example we can define and work with the *space* of triangles, or the *space* of configurations of a physical system.

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We will call this structure $\mathbb{RP}^2$, the *real projective space of dimension two*.

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Does it satisfy our axioms?

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3. There exist four *projective points*, no three of which are in a *projective line*.
$\mathbb{RP}^2$ satisfies this as well!
Consider the vectors $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and $(1,1,1)$ based at A.
No three of them will lie on the same plane through A.

Vision and the projective plane

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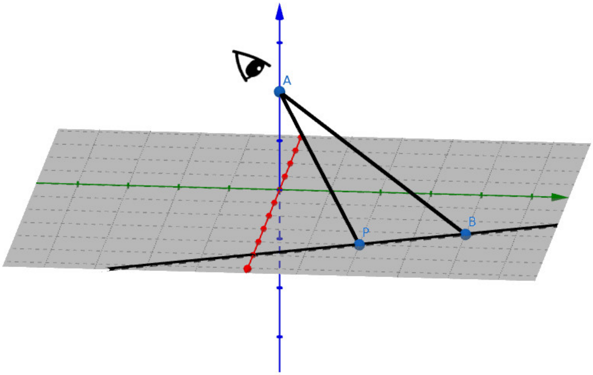
It captures the idea of viewing $\mathbb{R}^2$ with an all seeing eye based at A.



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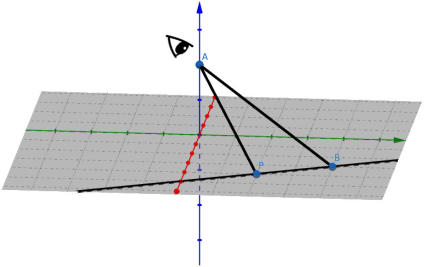

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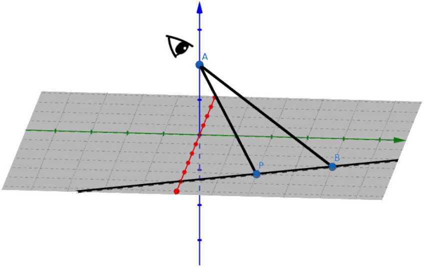
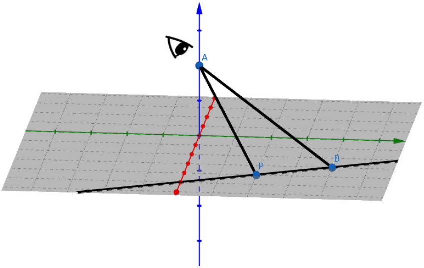
Vision and the projective plane



The lines through A can be thought of as lines of sight, which connect the eye to the points it sees.

In fact, all the points on the tiled xy-plane correspond to lines through A. Are there any remaining lines through A?



We have actually *extended* the xy plane to a *projective space* by adding 'points' at infinity (the horizontal lines through A) which together form a 'line' at infinity (the horizontal plane through A).

Our earlier perspective views

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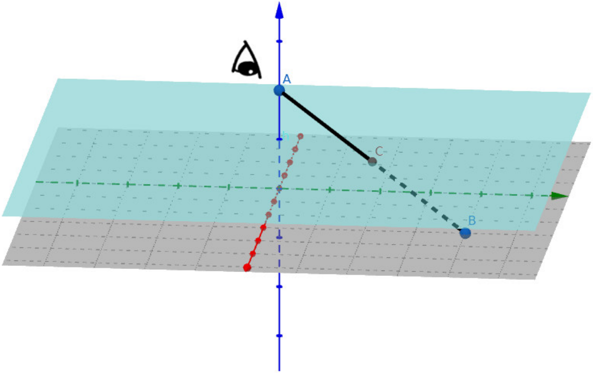
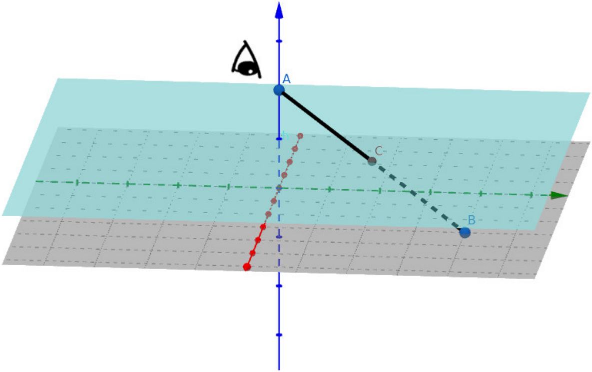


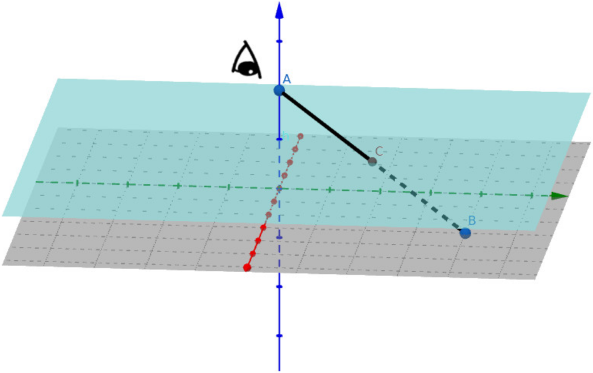
Our earlier perspective views

We can think of it as an (*affine*) projection of $\mathbb{RP}^2$.



Our earlier perspective views

So $\mathbb{RP}^2$ actually captures all possible perspective views at one time, before specializing to a given picture plane.



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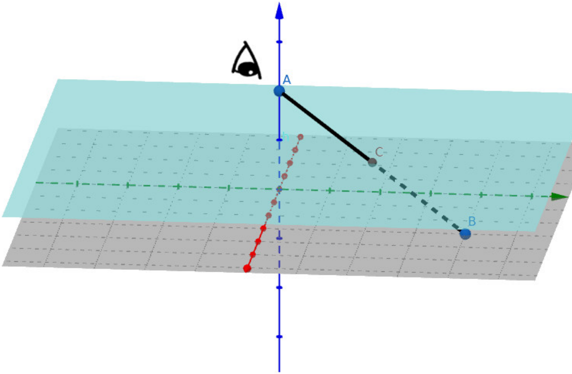

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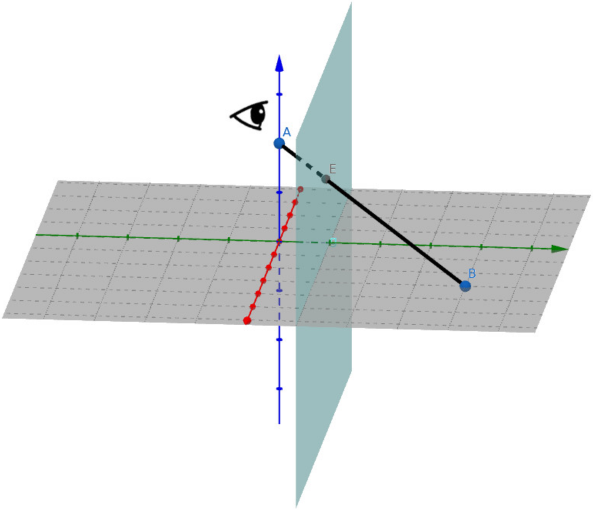

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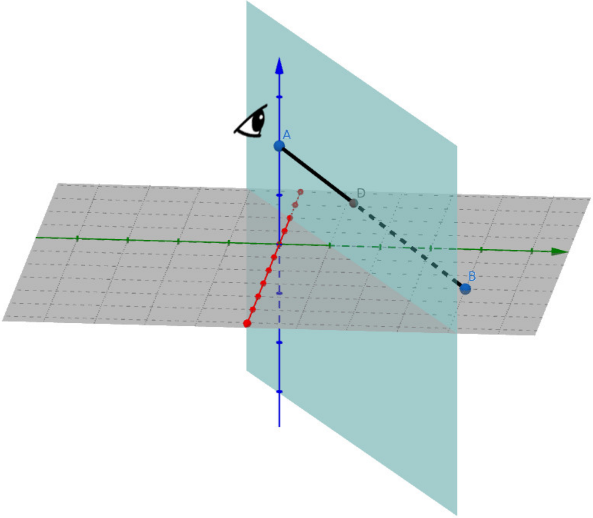

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A line through the origin is determined by a point (a, b, c) and consists of the points (ta, tb, tc) where t runs through all real numbers.

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It is easy to check that $(a, b, c) \sim (ta, tb, tc)$ is an equivalence relation. We will denote its equivalence class $[a : b : c]$, which we call the *homogeneous coordinates* of the corresponding *point* in $\mathbb{RP}^2$.

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A line through the origin is determined by a point (a,b,c) and consists of the points (ta,tb,tc) where t runs through all real numbers.

It is easy to check that $(a,b,c) \sim (ta,tb,tc)$ is an equivalence relation. We will denote its equivalence class $[a:b:c]$, which we call the *homogeneous coordinates* of the corresponding *point* in $\mathbb{RP}^2$.
One can make very precise statements about projective space using homogeneous coordinates, but we'll leave that for another day.

Back to the drawing board

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For example, we can easily draw boxes, houses, and even a whole city.

Back to the drawing board

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But there can be vanishing points off the horizon as well.

Back to the drawing board

Can you find vanishing points off of the horizon?

Convergence of parallels

Can we then explain why parallel lines converge, and where they converge to?

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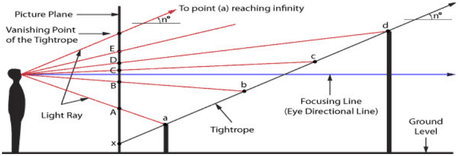

Using homogeneous coordinates

Here's another explanation:
Let $p = (p_1, p_2, p_3)$ be any point in $\mathbb{R}^3$ and let $v = (v_1, v_2, v_3)$ be any vector *not* parallel to the plane $z = 1$.

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Imagine we are looking further and further out along the line $L = \{p + tv : t \in \mathbb{R}\}$. At time t we see the *point* $[p_1 + tv_1 : p_2 + tv_2 : p_3 + tv_3] = [\frac{p_1+tv_1}{p_3+tv_3} : \frac{p_2+tv_2}{p_3+tv_3} : 1]$.

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Thus $(\frac{v_1}{v_3}, \frac{v_2}{v_3}, 1)$ is the vanishing point in the picture plane $z = 1$, not only of the line L, but of all lines parallel to it!

Further exploration

Much of the material for this talk came from *The Four Pillars of Geometry* by John Stillwell. If you're interested, you can read the book to learn more about

*projective geometry

*$\mathbb{RP}^2$

*the cross ratio (the actual quantity that is preserved under perspective transformations.)