Getting your head around spatial rotations

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CMI

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Meet Balla

This is Balla. A simple three-string puppet.

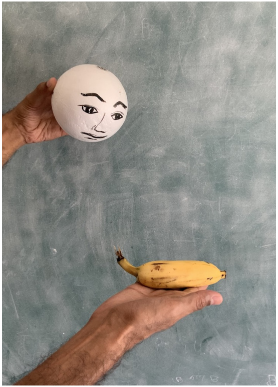
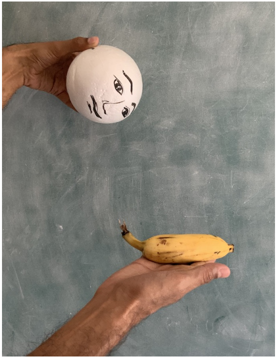
Meet Balla

But with just three strings, Balla can move in all kinds of ways! He can nod his head 'yes'. He can shake his head 'no'. And he can even say maybe so!

What can we learn from Balla?

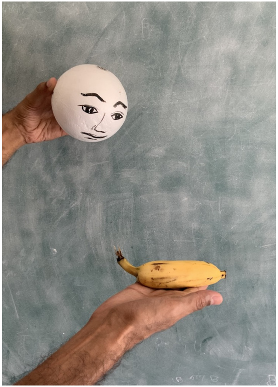
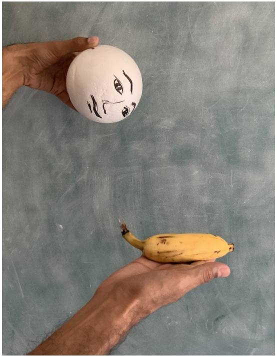
But Balla lost his strings, one by one, as he got older. And with each string lost, he also lost a degree of freedom of motion!

What is an oriention?


What are the possible spatial orientations of string-less Balla?
Balla's spatial orientation can be thought of as where he is facing and how his face is tilted.

What is an oriention?


What are the possible spatial orientations of string-less Balla?
It is not enough to say which direction his face is pointing. Balla can look at the banana via multiple spatial orientations, keeping his focus constant by rotating his head!

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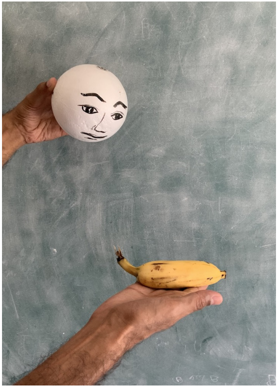 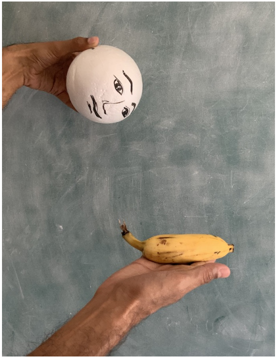 

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How many ways can Balla look at the banana, while keeping the center of his head-sphere fixed?

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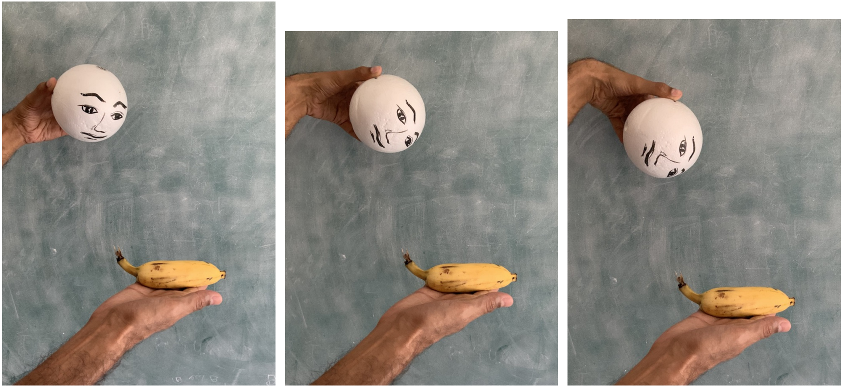
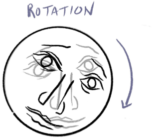


What are the possible spatial orientations of string-less Balla?
How many ways can Balla look at the banana, while keeping the center of his head-sphere fixed?
A circle's worth of ways! His head can rotate about the axis joining the center of his head to the banana.

Does position in space matter?



Rotating Balla about any axis changes his spatial orientation.

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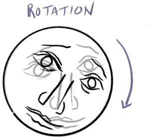

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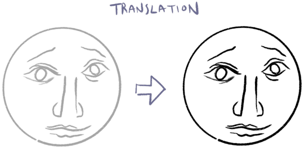
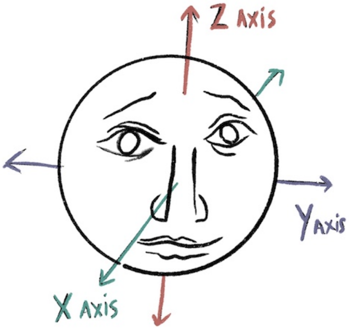


But translating Balla in space does not affect his spatial orientation. His spatial orientation does not care about position in space.

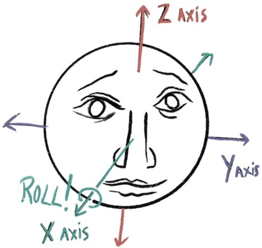
Balla's spatial orientations

What were the different spatial orientations he could realize, before his strings were cut?

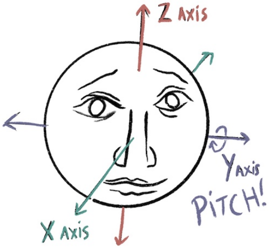
Three axes of rotation



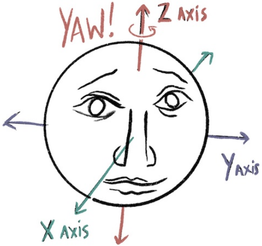
Rotating about X axis



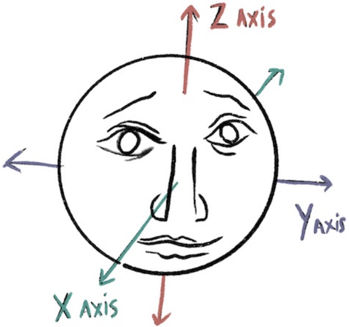
Rotating about Y axis



Rotating about Z axis



How many spatial orientations are there?



And now that he is string-less, are there any new spatial orientations he can find?
Can any spatial orientation be achieved through pitch, roll, and yaw?

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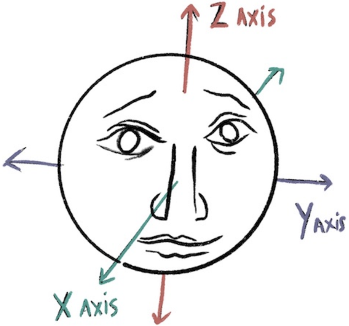
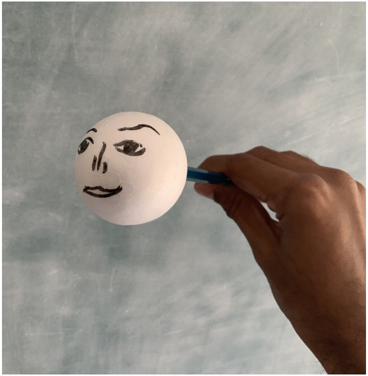


And now that he is string-less, are there any new spatial orientations he can find?

Can any spatial orientation be achieved through pitch, roll, and yaw?

Yes, rotations about the three axes were enough to arrive at ANY spatial orientation.

Exercise!



At the other extreme, do we really need *all* three types of rotations?

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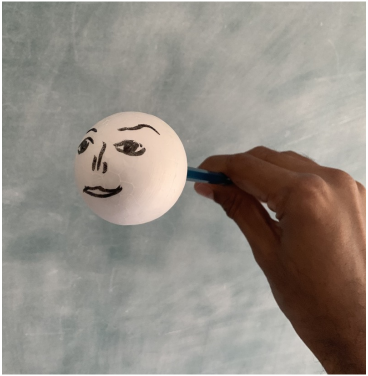
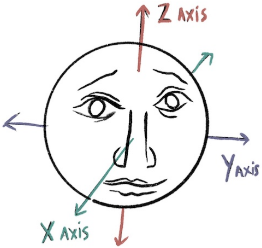


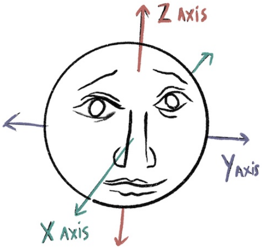
At the other extreme, do we really need *all* three types of rotations?
Can you find a relation between rotations with respect to these three axes?
Make your own puppet by drawing a face on the ball you received and check for yourself!

Solution



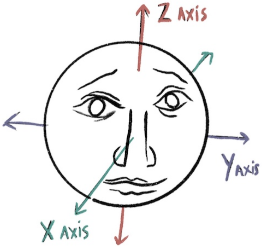
Here's one relation:

Solution



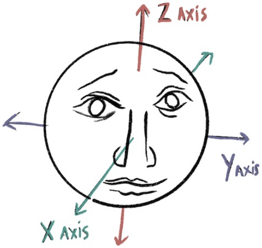
Here's one relation:
First perform a 90 degree rotation about the Y axis.
Next perform a 90 degree rotation about X axis.
Finally perform a -90 degree about Y axis.
This is equivalent to perfomring a 90 degree rotation about the Z axis.

Surprising?



This example shows two things:
Rotations do not commute!
And the three axes of rotation are not independent!

Two axes is enough!?



Hmm, so rotations about X and Y are enough to realize any spatial orientation? Maybe 90 degrees was special?

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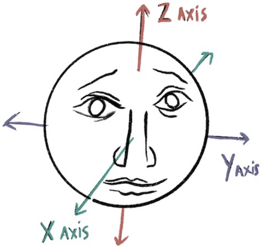

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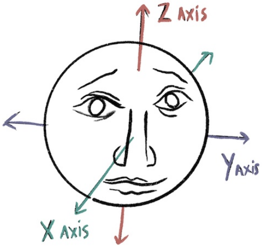

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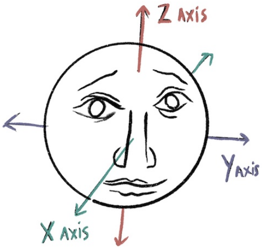
But that doesn't mean the set of spatial orientations is two dimensional...

Set of spatial orientations is three dimensional



No, the set of spatial orientations is indeed three dimensional. Three parameters are needed to specificy a spatial orientation.
An easy way to picture it is with an *axis* and an *angle*.
The axis corresponds to the direction the puppet is looking in. But then there are a circle's worth of rotations about that axis, which do not alter the direction of focus.

Spatial orientations = Rotations?

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They are in bijection! We can identify a spatial orientation with a single rotation that gets us there, from ourfixed starting orientation.
For this to make sense, we need a basic fact: all rigid motions of the sphere are rotations!

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But beware, the axis we used to refer to a spatial orientation is the direction the puppet is looking! Which is not the same as the axis of the unique rotation that got the puppet into this spatial orientation!

Euler's Theorem

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Rotations, right? Any rotation fixes exactly two antipodal points.
But Euler's Theorem tells us that all transformations are rotations, and must have fixed points!
No matter how you move a ball around, there must be a pair of antipodal points that are fixed!

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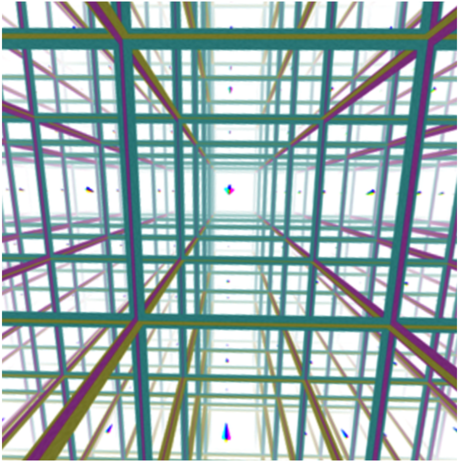
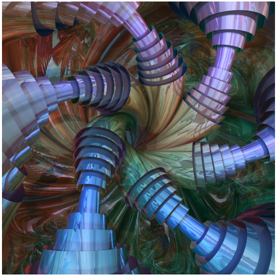
So any spatial orientation is in fact arrived at by a single rotation. No matter how Balla's head is positioned, a single rotation got him there!

A BIG Question



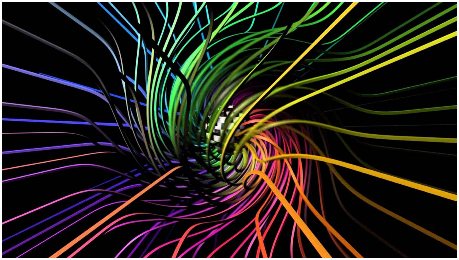
The set of spatial orientations is three dimensional.
But what does it 'look' like?
What shape is it?

Some three dimensional spaces



The euclidean space R^3
The hypersphere S^3
The hypertorus T^3
Could it be one of these?

No! It's something stranger still!



It is none of the above!

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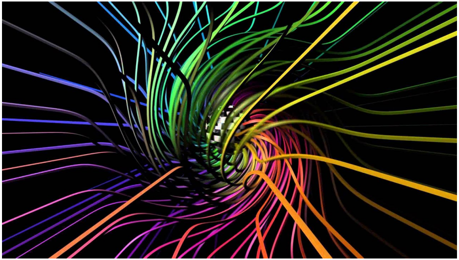

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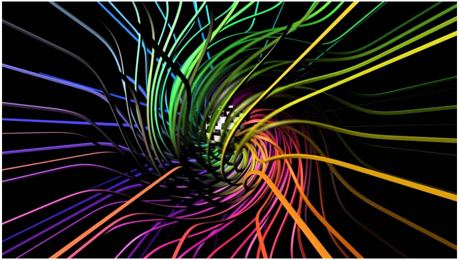

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Strange Fact: The set of spatial orientations has a loop that behaves very unexpectedly!
What is this loop? Well, any 720 degree rotation!

Plate trick

This special property allows you to do this.

Belt trick

Or this.

String trick

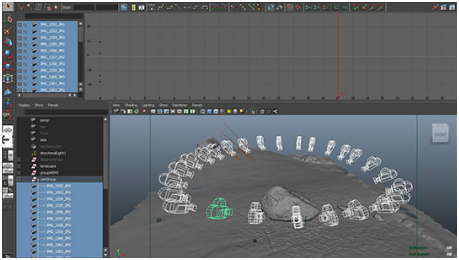
This is perhaps the strangest instance of this phenomenon.

You can connect any number of strings to an any object of your choice.

Rotate the object once and they get hopelessly tangled.

Rotate it again, and they untangle! Try it yourself!

Paths of spatial orientations???



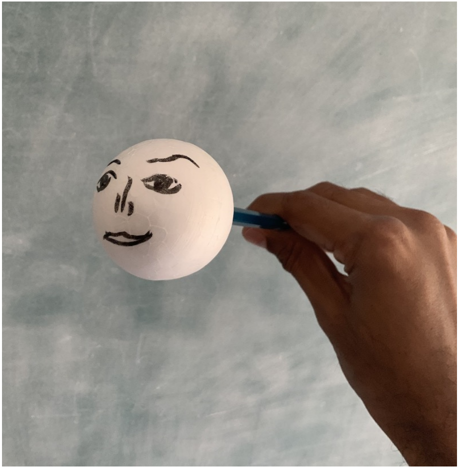
Let's back up a second.

We're talking about paths in the set of spatial orientations (aka rotations).

Trajectory of your gaze over time.

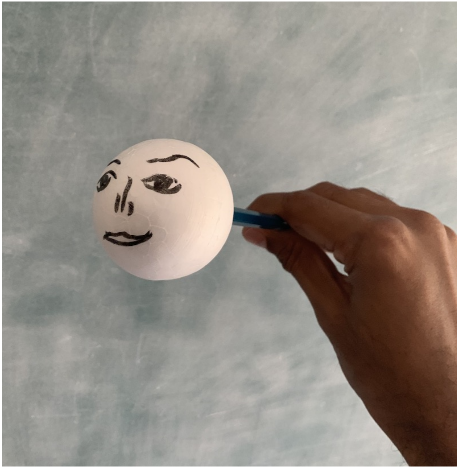
Or a camera path, as they call it in the film industry.

Let's try performing some paths!



Get your puppet ready! Stand up from your seat, and warm up your arms and shoulders! For this activity we're going to need to move around!

Warming up!



Are your arms warmed up? Good! Now find a simple 'path' you can perform with your puppet. Can you perform the same path with your own head?

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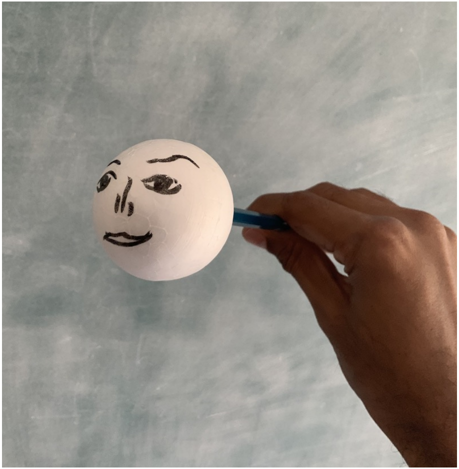
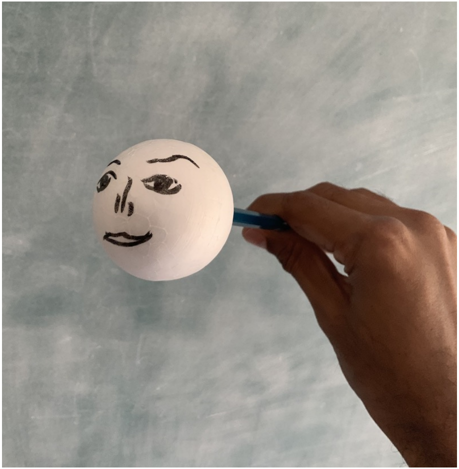


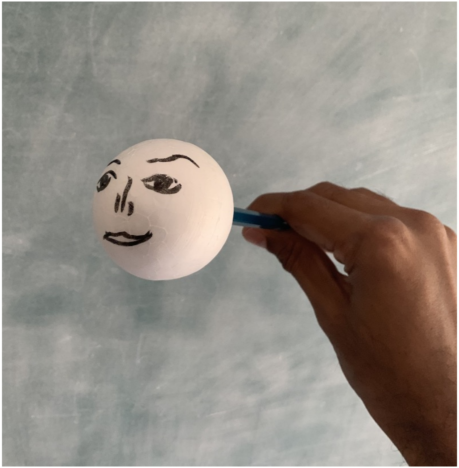
Try to find one path you can perform with both your own head and the puppet. And another path that you can perform with the puppet, but not with your own head!

Exerise 1: the stationary path



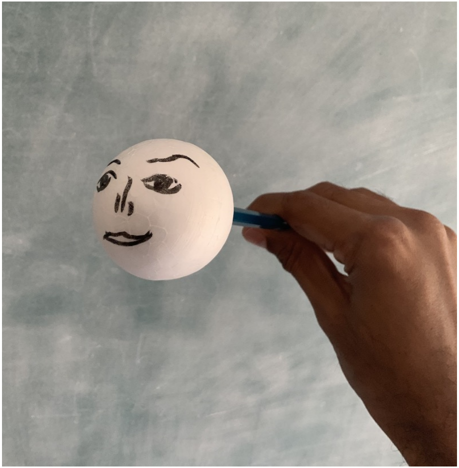
Now let's create the simplest path of all! Pick your favorite spatial orientation and HOLD IT, don't move at all!

Exercise 2: a wobble



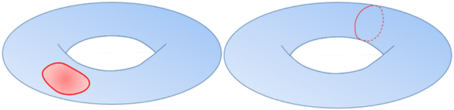
Try to make your puppet wobble in a small loop, so that its final spatial orientation is the same as its initial spatial orientation.

Exercise 3: a full 360 degree rotation loop



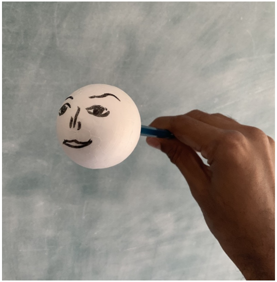
Can you make your puppet roll, tilt, and yaw, a full 360 degrees?

Trivial loops



We say a loop in any space is trivial if it can me smoothly deformed into the stationary path.
We can 'perform' the smooth deformation by repeating the loop over and over, with a minor adjustment each time, until we are back to the stationary loop.
The loop on the left is trivial, the loop on the right is nontrivial.

A trivial loop in the space of rotations

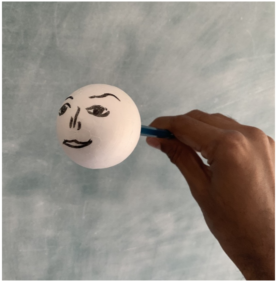


The wobble you did with your puppet is a trivial loop.
You can smoothly shrink it further and further until you get back to the stationary path.

Exercise 4: another trivial loop in the space of rotations

Consider a loop that does a full 360 degree rotation about an axis, followed by another full 360 degree rotation about the same axis, but in the opposite direction.
Can you show that this is trivial? Can you perform the smooth deformation?

A NONtrivial loop in the space of rotations

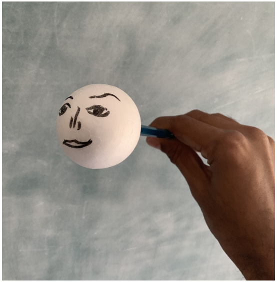


The full 360 degree rotation you did is a nontrivial loop.
You cannot smoothly deform it to the stationary path, no matter how hard you try!
But proving this requires some extra machinary, which you'll learn if you every take a course in Algebraic Topology.

Equivalent loops

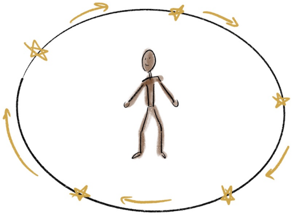
What happens if we smoothly deform a nontrivial loop?
We get another nontrivial loop!
We say all these loops are equivalent (or 'homotopic') to each other.

Exercise 5: any two 360 degree rotation loops are homotopic



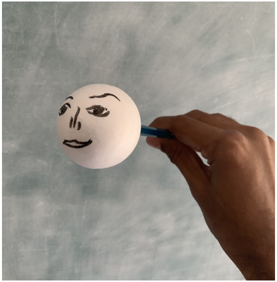
Here's a challenge!
Pick any two axes of rotation. Can you show that the 360 degree rotation loop about one axis is smoothly deformable to the 360 degree rotation loop about the other axis?
Can you perform the homotopy?

Exercise 6: two ways to track on object



Not bad! Here's an easier problem: imagine a particle orbiting about you in a vertical circle.
There are two very different loops in the set of spatial orientations that follow this particle as it moves, one trivial and one nontrivial.
Can you find both ways to track the particle?
Warning: only demonstrate the nontrivial path with your puppet, doing so with your own head will result in serious spinal injury!

Exercise 7: The Big One!



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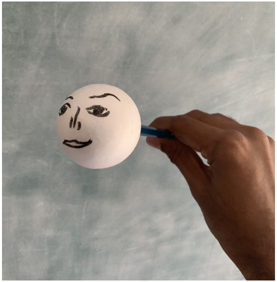

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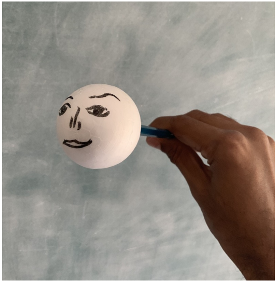

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Can you prove it? Demonstrate with your puppet how to smoothly deform a 720 degree rotation loop to the staionary path.

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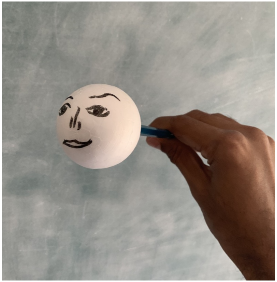

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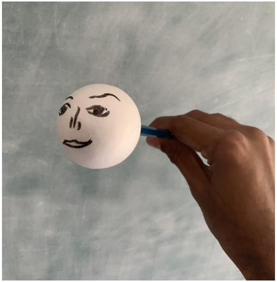
HINT: The figure eight is your friend!

Exercise 8: Coda



Putting all this together, here's some homework. Can you convince yourself that there are only two types of loops: trivial and nontrivial? In other words, all nontrivial loops are homotopic to each other!

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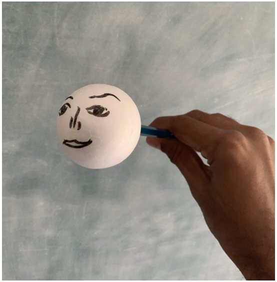

Putting all this together, here's some homework. Can you convince yourself that there are only two types of loops: trivial and nontrivial? In other words, all nontrivial loops are homotopic to each other!

Not convinced? Come to the stage and perform a loop a you're not sure about. We can figure out together if it is trivial or nontrivial!

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It sometimes goes by $SO(3)$, the special orthogonal group.
And when you're just concerned with its shape, it goes by $\mathbb{R}P^3$, the real projective space.

Time for some algebra?

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Can we capture these strange properties more precisely?

Is there a symbolic language for communicating things like paths and loops and homotopies in $SO(3)$? An *algebra* of the set of spatial orientations?

Rotation Matrices?

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Remember these?

Rotation Matrices?

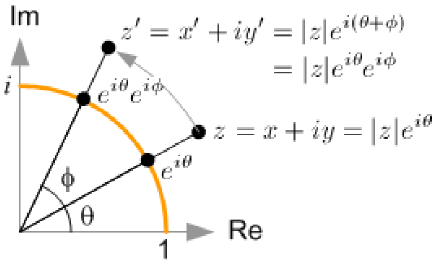
$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

How about these?
Rotation matrices have too many (nine!) dimensions.
Which suggests they are not the perfect tool.
On the other hand axis-angle representation is geometrically nice, but how on earth do we compose them?

What about Complex Numbers...



Let's think back to spatial orientations and rotations of $\mathbb{R}^2$.
Could there be a similarly elegant solution in $\mathbb{R}^3$?

Enter the Quaternions!

Quaternion multiplication

×	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

Yes indeed there is. The Quaternions.
Quaternions are a number system that extends the complex numbers.
A quaternion can be represented as $a + bi + cj + dk$ where where a, b, c, and d are real numbers, and i, j, and k are the fundamental quaternion units.

Unit quaternions up to sign

As in the complex case, spatial orientations correspond to *unit* quaternions. Unlike the complex case, we also need to identify unit quaternions that differ by a factor of -1.
Unit quaternions up to sign then correspond perfectly to spatial rotations of $\mathbb{R}^3$.

A bijectiion: unit quaternions and spatial orientations

Given an unit vector $(v_1, v_2, v_3) \in \mathbb{R}^3$ and an angle θ, the corresponding unit quaternion is
$\cos(\frac{\theta}{2}) + \sin(\frac{\theta}{2})(v_1 i + v_2 j + v_3 k)$.

In the other direction, given a unit quaternion $a + bi + cj + dk$, the corresponding spatial orientation/rotation is as follows:

angle: $\theta = 2\arccos(a)$

axis: $v = \frac{1}{\sin(\theta/2)}(b, c, d)$
The upshot is that i, j, k tell us the amount of rotation with respect to the x, y, z axes respectively!

A Quaternion App? For your phone?

This bijection is used all the time. For example, your phone contains a gyroscope, which calculates its spatial orientation at any moment. There are apps to translate this data into quaternions, which is in turn used by lots of common software.

Example 0: a point in SO(3)

What would a 90 degree rotation about the Z axis look like in the language of quaternions?

$\frac{\sqrt{2}}{2}(1 + k)$

Can you see why?

Remember: given an unit vector $(v_1, v_2, v_3) \in \mathbb{R}^3$ and an angle θ, the corresponding unit quaternion is $\cos(\frac{\theta}{2}) + \sin(\frac{\theta}{2})(v_1 i + v_2 j + v_3 k)$.

Example 1: a relation between points in SO(3)

Remember how rotating 90 degrees in the Z axis was equivalent to rotating in the Y axis, followed by X, followed by Y again?

In quaternion language, that's saying:

$\frac{\sqrt{2}}{2}(1-j)\frac{\sqrt{2}}{2}(1+i)\frac{\sqrt{2}}{2}(1+j)$

$= \frac{\sqrt{2}}{2}(1+k)$

An easy calculation you can check for youself!

Example 2: a path in SO(3)

What about a 360 degree rotation loop about the Y axis?

This path is parametrized by

$P(\theta) = \cos(\frac{\theta}{2}) + \sin(\frac{\theta}{2})j,$

as $0 \leq \theta \leq 2\pi$.

Example 3: a homotopy of nontrivial loops

Remember how any two 360 degree rotation loops are equivalent?
We can explicitly write the homotopy from the rotation loop about the X axis to the rotation loop about the Y axis:

$H(t, \theta) = \cos(\frac{\theta}{2}) + \sqrt{t}\sin(\frac{\theta}{2})i + \sqrt{1-t}\sin(\frac{\theta}{2})j,$

as $0 \leq \theta \leq 2\pi$.

Example 4: collapsing the 720 degree rotation

Exercise: Write this out explicitly!

Hint: First deform the 720 degree rotation loops into two 360 degree rotation loops in opposite directions!

Conclusion

Weird as they may seem, quaternions and spatial rotations are ubiquitous computer graphics, robotics, and many other areas of mechanical engineering.

But that's not all. The property we've studied is also present in all fermions (protons, neutrons, electrons), so strange as it sounds it is not at all obscure.

Further reading

Naive Lie Theory by John Stillwell

Wikipedia! And look at the list of references at the end!