PROBLEM SET 10

INTRO TO REAL ANALYSIS

Problem 1. Recall that a function $f : A \to \mathbb{R}$ is *Lipschitz* on A if there exists an $M > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M$$

for all $x \neq y$ in A.

(a) Show that if f is differentiable on a closed interval $[a, b]$ and if f' is continuous on $[a, b]$, then f is Lipschitz on $[a, b]$.

(b) The function f is *contractive* if there is a constant c such that $0 < c < 1$ and $|f(x) - f(y)| \leq c|x - y|$ for all $x, y \in A$. If we add the assumption that $|f'(x)| < 1$ on $[a, b]$, does it follow that f is contractive on this set?

Problem 2. Let f be differentiable on an interval A. If $f'(x) \neq 0$ on A, show that f is one-to-one on A. Provide an example to show that the converse statement need not be true.

Problem 3. A *fixed point* of a function f is a value x where $f(x) = x$. Show that if f is differentiable on an interval throughout which we have $f'(x) \neq 1$, then f can have at most one fixed point.

Problem 4. Suppose f and g are continuous functions on an interval containing a, and that f and g are differentiable on this interval (including at a). Moreover, suppose that $f(a) = g(a) = 0$, that f' and g' are continuous at a, and that $g'(x) \neq 0$ throughout the interval (including at a). Find a short proof that

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \text{ implies } \lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

Problem 5. Let $f : [a, b] \to \mathbb{R}$ be a one-to-one function that is both continuous and differentiable on $[a, b]$. Show that if $f'(x) \neq 0$ for all $x \in [a, b]$, then f^{-1} is differentiable on the range of f, with

$$(f^{-1})'(y) = \frac{1}{f'(x)} \text{ where } y = f(x).$$

*All questions taken from *Understanding Analysis: 2nd Edition* by Stephen Abbott.

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