PROBLEM SET 12

INTRO TO REAL ANALYSIS

Problem 1. Let
$$g(x) = \frac{nx + x^2}{2n},$$
and set $g(x) = \lim g_n(x)$. Show that g is differentiable in two ways:

(a) Compute g(x) by algebraically taking the limit as $n \to \infty$ and then find $g'(x)$.
(b) Compute $g_n'(x)$ for each $n \in \mathbb{N}$ and show that the sequence of derivatives (g_n') converges uniformly on every interval $[-M, M]$. Cite the appropriate theorem to conclude that $g'(x) = \lim g_n'(x)$.

Problem 2. Decide whether each proposition is true or false, providing a short justification or counterexample as appropriate.

(a) If $\sum_{n=1}^{\infty} g_n$ converges uniformly, then (g_n) converges uniformly to zero.
(b) If $0 \leq f_n(x) \leq g_n(x)$ and $\sum_{n=1}^{\infty} g_n$ converges uniformly, then $\sum_{n=1}^{\infty} f_n$ converges uniformly.
(c) If $\sum_{n=1}^{\infty} f_n$ converges uniformly on A, then there exist constants M_n such that $|f_n(x)| \leq M_n$ for all $x \in A$ and $\sum_{n=1}^{\infty} M_n$ converges.

Problem 3. (a) Prove that
$$h(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} = x + \frac{x^2}{4} + \frac{x^3}{9} + \cdots$$
is continuous on $[-1, 1]$.

(b) Note that the series
$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$
converges for every x in the interval $[-1, 1)$ but does not converge when $x = 1$. For a fixed $x_0 \in (-1, 1)$, explain how we can still use the Weierstrass M-Test to prove that f is continuous at x_0.

Problem 4. Let
$$f(x) = \frac{1}{x} - \frac{1}{x+1} + \frac{1}{x+2} - \frac{1}{x+3} + \cdots .$$
Show that f is defined for all $x > 0$. Is f continuous on $(0, \infty)$? Is f differentiable?

Problem 5. Let $\{r_1, r_2, \ldots\}$ be an enumeration of the set of rational numbers. For each $r_n \in \mathbb{Q}$, define
$$u_n(x) = \begin{cases} 1/2^n & \text{for } x > r_n \\ 0 & \text{for } x \leq r_n. \end{cases}$$
Let $h(x) = \sum_{n=1}^{\infty} u_n(x)$. Prove that h is a monotone function defined on all of $\mathbb{R}$ that is continuous at every irrational point.

*All questions taken from *Understanding Analysis: 2nd Edition* by Stephen Abbott.

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