# PROBLEM SET 1

INTRO TO REAL ANALYSIS

**Problem 1.** Prove that $\sqrt{3}$ is irrational. Does a similar argument work to show that $\sqrt{6}$ is irrational? Where does your proof break down for $\sqrt{9}$?

**Problem 2.** Show that there is no rational number $r$ satisfying $2^r = 3$.

**Problem 3.** Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific counterexample.

(a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all sets containing an infinite number of elements, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.

(b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and nonempty.

(c) $A \cap (B \cup C) = (A \cap B) \cup C$.

(d) $A \cap (B \cap C) = (A \cap B) \cap C$.

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

**Problem 4.** Produce an infinite collection of sets $A_1, A_2, A_3, \ldots$ with the property that every $A_i$ has an infinite number of elements, $A_i \cap A_j = \varnothing$ for all $i \neq j$, and $\bigcup_{i=1}^{\infty} A_i = \mathbb{N}$.

**Problem 5** (De Morgan's Laws)**.** Let $A$ and $B$ be subsets of $\mathbb{R}$.

(a) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subseteq A^c \cup B^c$.

(b) Prove the reverse inclusion $(A \cap B)^c \supseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$.

(c) Prove that $(A \cup B)^c = A^c \cap B^c$.

**The following problem is optional. It will not contribute to or detract from your grade, but you are encouraged to attempt it.**

**Challenge 1.** (a) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cdots \cap A_n^c$$

for any finite $n \in \mathbb{N}$.

(b) It is tempting to appeal to induction to conclude

$$\left( \bigcup_{i=1}^{\infty} A_i \right)^c = \bigcap_{i=1}^{\infty} A_i^c,$$

but induction does not apply here. In general, induction can only prove that a particular statement holds for every value of $n \in \mathbb{N}$, but this does not necessarily imply the infinite case. To illustrate this point, find a collection of sets $B_1, B_2, B_3, \ldots$ where the statement $\bigcap_{i=1}^{n} B_i \neq \varnothing$ is true for every $n \in \mathbb{N}$, but the statement $\bigcap_{i=1}^{\infty} B_i \neq \varnothing$ fails.

(c) Nevertheless, the infinite version of De Morgan's Law stated in $(b)$ is a valid statement. Provide a proof that does not use induction.

*All questions taken from *Understanding Analysis: 2nd Edition* by Stephen Abbott.

*Date*: August 7, 2017.