PROBLEM SET 5

INTRO TO REAL ANALYSIS

Problem 1. Let (a_n) and (b_n) be Cauchy sequences. Decide whether each of the following sequences is a Cauchy sequence, justifying each conclusion:
(a) $c_n = |a_n - b_n|$
(b) $c_n = (-1)^n a_n$
(c) $c_n = [[a_n]]$, where $[[x]]$ refers to the greatest integer less than or equal to x.

Problem 2. In the previous problem set, you established the equivalence of the Axiom of Completeness and the Monotone Convergence Theorem (make sure you understand why). You also showed that the Nested Interval Property is equivalent to these other two in the presence of the Archimedean Property.
(a) Assume the Bolzano-Weierstrass Theorem is true, and use it to prove the Monotone Convergence Theorem without making any appeal to the Archimedean Property.
(b) Use the Cauchy Criterion to prove the Bolzano-Weierstrass Theorem, and find the point in the argument where the Archimedean Property is implicitly required.
(c) How do we know it is impossible to prove the Axiom of Completeness starting from the Archimedean Property?

Problem 3. Let (a_n) be a sequence satisfying $(a_n) \to 0$, and also $a_k \geq a_{k+1}$ for all $k \in \mathbb{N}$. The Alternating Series Test (AST) states that $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges. Let $s_n = a_1 - a_2 + a_3 - \cdots \pm a_n$.
(a) Prove AST by showing that (s_n) is a Cauchy sequence.
(b) Give another proof of AST using the Nested Interval Property.
(c) Supply yet another proof of AST using the Monotone Convergence Theorem, and by considering the subsequences (s_{2n}) and (s_{2n+1}).

Problem 4. Use the Cauchy Condensation Test to prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

Problem 5. Given a series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$, the Ratio Test states that if (a_n) satisfies

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = r < 1,$$

then the series converges absolutely.
(a) Let r' satisfy $r < r' < 1$. Explain why there exists an N such that $n \geq N$ implies $|a_{n+1}| \leq |a_n| r'$.
(b) Why does $a_N \sum (r')^n$ converge?
(c) Show that $\sum |a_n|$ converges, and conclude that $\sum a_n$ converges.

*All questions taken from *Understanding Analysis: 2nd Edition* by Stephen Abbott.

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