PROBLEM SET 8

INTRO TO REAL ANALYSIS

Problem 1. Assume f and g are defined on all of $\mathbb{R}$ and that $\lim_{x\to p} f(x) = q$ and $\lim_{x\to q} g(x) = r$.
(a) Give an example to show that it may not be the case that

$$\lim_{x\to p} g(f(x)) = r.$$

(b) Show that the result in (a) does follow if we assume f and g are continuous.
(c) Does the result in (a) hold if we only assume f is continuous? What if we only assume g is continuous?

Problem 2. Show whether or not the following functions are *uniformly* continuous on the interval $(0, 1)$.
(a) $f(x) = 1/x$.
(b) $g(x) = \sqrt{x^2 + 1}$.
(c) $h(x) = x \sin(1/x)$.

Problem 3. A function $f : A \to \mathbb{R}$ is called *Lipschitz* if there exists a bound $M > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M$$

for all $x \neq y \in A$. In other words, there is a uniform bound on the magnitude of the slopes of lines drawn through any two points on the graph of f.
(a) Show that if $f : A \to \mathbb{R}$ is Lipschitz, then it is uniformly continuous on A.
(b) Does the converse hold? I.e. are uniformly continuous functions necessarily Lipschitz?

Problem 4. Let f be a continous one-to-one function from an interval A to $\mathbb{R}$.
(a) Show that f is monotone.
(b) Show that f^{-1} is continuous.

The following problem is optional. It will not contribute to or detract from your grade, but you are encouraged to attempt it.

Challenge 1. (a) Let g be defined on all of $\mathbb{R}$. Show that g is continuous if and only if $g^{-1}(U)$ is open whenever $U \subset \mathbb{R}$ is open.
(b) Let f be a continuous function defined on all of $\mathbb{R}$. Suppose B is a set with property $\spadesuit \in \{$finite, compact, bounded, closed$\}$. Does $g^{-1}(B)$ have property $\spadesuit$?

*All questions taken from *Understanding Analysis: 2nd Edition* by Stephen Abbott.

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