# PROBLEM SET 9

INTRO TO REAL ANALYSIS

**Problem 1.** Recall the Chain Rule: Suppose that $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ satisfy $f(A) \subset B$, that $f$ is differentiable at $c \in A$, and that $g$ is differentiable at $f(c) \in B$. Then it follows that $g \circ f$ is differentiable at $c$ with $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.

(a) Show that a function $h : A \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}$ if and only if there exists a function $l : A \to \mathbb{R}$ which is continuous at $a$ and satisfies

$$h(x) - h(a) = l(x)(x - a) \text{ for all } x \in A.$$

(b) Use the above criterion for differentiability to prove the Chain Rule (you will need to use both directions of the criterion).

**Problem 2.** Let

$$f_a(x) = \begin{cases} x^a & \text{if } x > 0 \\ 0 & \text{if } x \leq 0, \end{cases}$$

for all real numbers $a$.

(a) For which values of $a$ is $f$ continuous at zero?
(b) For which values of $a$ is $f$ differentiable at zero? In this case, is the derivative function continuous?
(c) For which values of $a$ is $f$ twice-differentiable?

**Problem 3.** Let

$$g_a(x) = \begin{cases} x^a \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

In each part, find one particular value for the number $a$ satisfying the given conditions. Namely, such that

(a) $g_a$ is differentiable on $\mathbb{R}$ but such that $g_a'$ is unbounded on $[0, 1]$.
(b) $g_a$ is differentiable on $\mathbb{R}$ with $g_a'$ continuous but not differentiable at zero.
(c) $g_a$ is differentiable on $\mathbb{R}$ and $g_a'$ is differentiable on $\mathbb{R}$, but such that $g_a''$ is not continuous at zero.

*All questions taken from *Understanding Analysis: 2nd Edition* by Stephen Abbott.

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