# PROBLEM SET 4

ANALYSIS II

**Problem 1.** Prove that any linear transformation from $\mathbb{R}^m \to \mathbb{R}^n$ is continuous.

**Problem 2.** Suppose $f$ is a function from a metric space $(X, d_X)$ to a metric space $(Y, d_Y)$. Prove that $f$ is continuous if and only if $f^{-1}(U)$ is open in $X$ for every open set $U$ of $Y$.

**Problem 3.** (a) Show that if $Q$ is a rectangle, then $Q$ equals the closure of $\text{Int}Q$.
(b) If $D$ is a closed set, what is the relation in general between the set $D$ and the closure of $\text{Int}D$?
(c) If $U$ is an open set, what is the relation in general between the set $U$ and the interior of the closure of $U$?

**Problem 4.** Let $A = \{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } 0 < y < x^2\}$.
(a) Show that every straight line through $(0, 0)$ contains an interval around $(0, 0)$ which is in $\mathbb{R}^2 \setminus A$.
(b) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \notin A, \text{ and} \\ 1 & \text{if } x \in A. \end{cases}$$

For $h \in \mathbb{R}^2$ define $g_h : \mathbb{R} \to \mathbb{R}$ by $g_h(t) = f(th)$. Show that each $g_h$ is continuous at 0, but $f$ is not continuous at $(0, 0)$.

**Problem 5.** Let $\mathbb{R}^\infty = \bigcup_{n=1}^\infty \mathbb{R}^n$ where we consider the natural inclusions $\mathbb{R}^n \subset \mathbb{R}^{n+1}$. (You can also think of this as those elements of $\mathbb{R}^\omega$ with finitely many nonzero entries.) Note that the dot product gives a well-defined inner product on $\mathbb{R}^\infty$, and hence induces a metric. Define $e_i = (0, 0, \ldots, 0, 1, 0, 0, \ldots)$ where 1 appears in the $i^{\text{th}}$ place. Prove that $X := \{e_i : i \in \mathbb{N}\}$ forms a basis for $\mathbb{R}^\infty$, and that $X$ is closed, bounded, and noncompact.

**Challenge 1.** Let $(X, d)$ be a metric space. Show that a subset $A \subset X$ is compact (i.e. every open cover of $A$ has a finite subcover) if and only if $A$ is sequentially compact (i.e. every sequence in $A$ has a subsequence that converges to a point in $A$).

*All questions taken from *Analysis on Manifolds* by James Munkres and *Calculus on Manifolds* by Michael Spivak.

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