PROBLEM SET 5

ANALYSIS II

Problem 1. Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(0,0) = 0$ and $f(x,y) = h(x,y)$ for one of the following expressions h when $(x,y) \neq (0,0)$. Analyze the behavior of each function f at $\mathbf{0} = (0,0)$. In particular, determine for which vectors $\mathbf{u}$ the directional derivative $f'(\mathbf{0}, \mathbf{u})$ exists, evaluate it for those $\mathbf{u}$, determine $Df(\mathbf{0})$ (if it exists), and determine whether f is continuous at $\mathbf{0}$.

(a) $\frac{xy}{x^2+y^2}$

(b) $\frac{x^2y^2}{x^2y^2+(y-x)^2}$

(c) $\frac{x^3}{x^2+y^2}$

(d) $|x| + |y|$

(e) $|xy|^{1/2}$

(f) $\frac{x|y|}{\sqrt{x^2+y^2}}$

Problem 2. For each of the following functions $f : \mathbb{R}^n \to \mathbb{R}^n$, determine Df and $\det Df$, and sketch the image of the given set S under f.

(a) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $f(r,\theta) = (r\cos\theta, r\sin\theta)$. Let $S = [1,2] \times [0,\pi]$.

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $f(x,y) = (x^2-y^2, 2xy)$. Let $S = \{(x,y) : x^2+y^2 \leq a, x \geq 0, y \geq 0\}$, for some positive real number a.

(c) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $f(x,y) = (e^x\cos y, e^x \sin y)$. Let $S = [0,1] \times [0,\pi]$.

(d) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$, defined by $f(\rho,\phi,\theta) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$. Let $S = [1,2] \times [0,\pi/2] \times [0,\pi/2]$.

*All questions taken from *Analysis on Manifolds* by James Munkres and *Calculus on Manifolds* by Michael Spivak.

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