PROBLEM SET 6

ANALYSIS II

Problem 1. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be given by the equation $f(x) = ||x||^2 \cdot x$. Show that f is of class C^∞ and that f carries the unit ball $U(0;1)$ onto itself in a one-to-one fashion. Show, however, that the inverse function is not differentiable at 0.

Problem 2. Let A be an open subset of $\mathbb{R}^n$, and let $H : A \to \mathbb{R}^n$ be a function of class C^1 such that $DH(a) = \mathbf{0}$ for some $a \in A$. Prove that for any $\epsilon > 0$, there exists a neighborhood U of a such that

$$\frac{||H(x_0) - H(x_1)||}{||x_0 - x_1||} < \epsilon$$

for any pair of points x_0 and x_1 in U.

Problem 3. Let A be open in $\mathbb{R}^n$. Let $f : A \to \mathbb{R}^n$ be of class C^r. Assume $Df(x)$ is nonsingular for $x \in A$. Show that even if f is not one-to-one on A, the set $B = f(A)$ is open in $\mathbb{R}^n$.

Problem 4. Given $f : \mathbb{R}^5 \to \mathbb{R}^2$, of class C^1. Let $a = (1, 2, -1, 3, 0)$. Suppose that $f(a) = 0$ and

$$Df(a) = \begin{bmatrix} 1 & 3 & 1 & -1 & 2 \\ 0 & 0 & 1 & 2 & -4 \end{bmatrix}.$$

(a) Show there is a function $g : B \to \mathbb{R}^2$ of class C^1 defined on an open set B of $\mathbb{R}^3$ such that

$$f(x_1, g_1(x), g_2(x), x_2, x_3) = 0$$

for $x = (x_1, x_2, x_3) \in B$, and $g(1, 3, 0) = (2, -1)$.

(b) Find $Dg(1, 3, 0)$.

Problem 5. Let $f : \mathbb{R}^{k+n} \to \mathbb{R}^n$ be of class C^1. Suppose $f(a) = 0$ and that $Df(a)$ has rank n. Show that if c is a point of $\mathbb{R}^n$ sufficiently close to 0, then the equation $f(x) = c$ has a solution.

*Questions taken from *Analysis on Manifolds* by James Munkres and *Calculus on Manifolds* by Michael Spivak.

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