PROBLEM SET 8

ANALYSIS II

Problem 1. (a) Given a symmetric $m \times m$ matrix A, let $Q_A : \mathbb{R}^m \to \mathbb{R}$ be defined by $x \mapsto x^t Ax$. Show that $\nabla Q_A(x) = 2Ax$, for any $x \in \mathbb{R}^m$.

(b) Let $\phi : \mathbb{R}^m \to \mathbb{R}$ be a function of class C^2. Let x be a critical point of ϕ (i.e. suppose $\nabla\phi(x) = 0$). Let

$$H(x) = \left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}(x)\right)$$

be the Hessian of ϕ at x. In class we defined $\frac{\partial^2 \phi}{\partial v^2}(x)$ to be $\frac{d^2}{dt^2}[\phi(x+tv)]_{t=0}$. Prove that

$$\frac{\partial^2 \phi}{\partial v^2}(x) = v^t H(x) v$$

for any $v \in \mathbb{R}^m$. (Hint: it will be useful to prove that $\frac{\partial^2 \phi}{\partial v^2}(x) = \frac{\partial}{\partial v}(\frac{\partial}{\partial v}\phi)(x)$.)

Problem 2. Prove the *second order Taylor's theorem* discussed in class. Namely, let $B = U(a, \epsilon) \subset \mathbb{R}^m$ be an open ball centered at a point a in $\mathbb{R}^m$, and suppose $f : B \to \mathbb{R}$ is of class C^2. Show that for any $x \in B$, we have

$$f(x) = f(a) + (x-a)^t \nabla f(a) + \frac{1}{2}(x-a)^t H(b)(x-a),$$

for some b on the line segment joining a to x, where $H(b)$ denotes the Hessian of f at b.

Problem 3. In class we defined the *operator norm* on the set of $m \times m$ matrices by

$$||A|| := \sup\{||Ax||/||x|| : x \in \mathbb{R}^m\} = \sup\{||Ax|| : x \in S^{m-1}\},$$

where $||v||$ denotes the Euclidean norm on $v \in \mathbb{R}^m$. Prove that:

(a) The operator norm is indeed a norm.

(b) The operator norm is a *continuous* function from $\mathbb{R}^{m^2}$ to R (here, we identify the set of $m \times m$ matrices with the Euclidean space $\mathbb{R}^{m^2}$).

Problem 4. Suppose A is a symmetric $m \times m$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_m$. Moreover, suppose the eigenvalues are indexed so that $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_m|$. Show that $||A|| = |\lambda_1|$.

Problem 5. Prove the second derivative test. Namely, let $U \subset \mathbb{R}^m$ be an open set. Suppose $\phi : U \to \mathbb{R}$ is of class C^2, and that $\nabla\phi(x) = 0$ for some x in U. Show that if $\frac{\partial^2 \phi}{\partial v^2}(x) > 0$ for any $v \in \mathbb{R}^m$ then ϕ attains a local minimum at x. (By *local minimum*, we mean that ϕ takes strictly greater values in some deleted neighborhood of x.)

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