TUTORIAL 1

ANALYSIS II

I. If you haven't already:

Problem 1. Let f be a bounded function on an interval $[a, b]$ and let c be a point in (a, b). Prove that f is integrable on $[a, b]$ if and only if f is integrable on both $[a, c]$ and $[c, b]$.

II. Integrability and sequences of functions:

Problem 2. Provide an example or explain why the request is impossible:

(a) A sequence $(f_n) \to f$ pointwise where each f_n has (at most) finitely many discontinuities but f is not integrable.

(b) A sequence $(g_n) \to g$ uniformly where each g_n has finitely many discontinuities but g is not integrable.

(c) A sequence $(h_n) \to h$ uniformly where each h_n is not integrable but h is integrable.

III. Integrability of some pathological functions:

Problem 3. Consider Thomae's function $t(x)$ on the interval $[0, 1]$:

$$t(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \in \mathbb{Q} \setminus \{0\} \text{ is in lowest terms} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Recall that $t(x)$ is continuous at all irrational points and has discontinuities at all rational points (why is this?). Show that $t(x)$ is integrable on $[0, 1]$.

Problem 4. Let C denote the Cantor set in $[0, 1]$. Define a Cantor function on $[0, 1]$ by

$$h(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \notin C. \end{cases}$$

The function $h(x)$ is continuous on all points not in C, and has discontinuities at all points in C (why is this?). Show that $h(x)$ is integrable on $[0, 1]$.

IV. The measure of the set of discontinuities:

Let E be a subset of $\mathbb{R}$. We define the *outer measure* of E to be

$$\lambda^*(E) = \inf\{\sum_{n=1}^{\infty} \ell(I_n) : (I_n) \text{ is a sequence of closed intervals covering } E\}.$$

We say that a set $E \subset \mathbb{R}$ has *measure zero* if $\lambda^*(E) = 0$: i.e. if for any $\epsilon > 0$ there exists a sequence of closed intervals covering E with total length less than ϵ.

Problem 5. Show that any countably infinite set has measure zero.

Problem 6. Show that the Cantor set has measure zero.

*All questions taken from *Understanding Analysis: 2nd Edition* by Stephen Abbott.

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