TUTORIAL 2

ANALYSIS II

Problem 1. Let x and y be vectors in $\mathbb{R}^n$.

(a) For which values of x and y do we have $||x+y|| = ||x|| + ||y||$?
(b) For which values of x and y do we have $||x+y||^2 = ||x||^2 + ||y||^2$?

Problem 2. For each of the following matrices,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix},$$

(a) Compute the reduced row echelon form, and determine the rank of the matrix.
(b) Give an example of a kernel element of the associated map T_A.
(c) Find bases for $\mathbb{R}^m$ and $\mathbb{R}^n$ so that the matrix representation of T_A is

$$\begin{bmatrix} \mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

for some p.

Problem 3. Let T be a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$, and consider the usual Euclidean inner product on both spaces.

(a) Prove that T is norm preserving (i.e. $||T(x)|| = ||x||$ for all $x \in V$) if and only if T is inner product preserving (i.e. $\langle T(x), T(y)\rangle = \langle x, y\rangle$ for all $x, y \in V$).
(b) Prove that T must be $1-1$ in this case.

Problem 4. Recall that in Euclidean space, the angle $\theta(x, y)$ between nonzero vectors x and y is related to their inner product by

$$\cos(\theta(x,y)) = \frac{\langle x, y\rangle}{||x|| \cdot ||y||}.$$

We say a linear transformation T is *angle preserving* if it is $1-1$ and $\theta(Tx, Ty) = \theta(x, y)$ for all nonzero vectors x and y.

(1) Prove that if T is norm preserving, then it is angle preserving.
(2) Give an example of an angle preserving map that is not norm preserving.
(3) Describe all angle preserving maps.

Problem 5. If $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation, show that there is a number M such that $||T(x)|| \leq M||x||$ for all $x \in \mathbb{R}^m$.

Problem 6. Let X denote $\mathbb{R}^2$ endowed with the Euclidean metric, and let Y denote $\mathbb{R}^2$ endowed with the sup metric. Let U be a subset of $\mathbb{R}^2$. Prove that U is an open set in X if and only if U is an open set in Y. (This shows that X and Y induce the same *topology* on $\mathbb{R}^2$).

Problem 7. Let $\mathbb{R}^\infty = \bigcup_{n=1}^\infty \mathbb{R}^n$ where we consider the natural inclusions $\mathbb{R}^n \subset \mathbb{R}^{n+1}$. (You can also think of this as those elements of $\mathbb{R}^\omega$ with finitely many nonzero entries.) Note that the dot product gives a well-defined inner product on $\mathbb{R}^\infty$, and hence induces a metric. Define $e_i = (0, 0, \ldots, 0, 1, 0, 0, \ldots)$ where 1 appears in the i^{th} place. Prove that $X := \{e_i : i \in \mathbb{N}\}$ forms a basis for $\mathbb{R}^\infty$, and that X is closed, bounded, and noncompact.

*All questions taken from *Analysis on Manifolds* by James Munkres and *Calculus on Manifolds* by Michael Spivak.

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