TUTORIAL 3

ANALYSIS II

Problem 1. Let $f : \mathbb{R}^n \supset A \to \mathbb{R}^m$, and let a be a limit point of A. Show that $\lim_{x\to a} f(x) = \mathbf{0}$ if and only if $\lim_{x\to a} ||f(x)|| = 0$. In particular, this shows that a function f is differentiable at $a \in \text{Int}(A)$ if and only if there exists an $n \times m$ matrix B such that

$$\lim_{h\to a} \frac{||f(a+h) - f(a) - B \cdot h||}{||h||} = 0.$$

Problem 2. Given a function $f : \mathbb{R}^n \to \mathbb{R}$, we can interpret $Df(x_1, \dots, x_n)$ as a vector field in $\mathbb{R}^n$. (This is known as the *gradient* vector field of f, and it points in the direction of steepest increase of the function f.) Sketch the gradient vector fields of the following functions:

(a) $f(x, y) = -xy$
(b) $f(x, y) = x^2 + y^2$

Problem 3. Let $A \subset \mathbb{R}^m$. Show that if the maximum (or minimum) of a function $f : A \to \mathbb{R}$ occurs at a point $a \in \text{Int}A$ and if $D_i f(a)$ exists, then $D_i f(a) = 0$.

Problem 4. Sketch the graphs of the following surfaces. Check if the functions attain any maximum or minimum values.

(a) $f(x, y) = y^2 + x^2$.
(b) $f(x, y = y^2 - x^2$.

Problem 5. Let $f : \mathbb{R} \to \mathbb{R}^3$ be defined by $f(t) = (t, t^2, t^3)$.

(a) Sketch the image of f in $\mathbb{R}^3$.
(b) Determine $Df(t)$. We can interpret this 3×1 matrix as giving the velocity vector of a parametrized curve at time t. Plot several velocity vectors in your sketch from part (a).
(c) Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ denote the projection onto the last two coordinates. Sketch the image of $\pi \circ f$ in $\mathbb{R}^2$, and calculate the derivative of this function.
(d) Can you interpret your sketch in part (c) as the graph of a function from $\mathbb{R} \to \mathbb{R}$ (perhaps after rotating it)? Calculate the derivative of this function as well.

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