TUTORIAL 4

ANALYSIS II

Problem 1. Prove that $\sigma : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is a bilinear if and only if it is of the form $(x, y) \mapsto x^t Ay$ for some $m \times m$ matrix A.

Problem 2. Let A be a symmetric $m \times m$ matrix. We say that A is *positive definite* if all its eigenvalues are positive.

(a) Prove that A is positive definite if and only if $x^t Ax > 0$ for all $x \neq 0$ (i.e. if and only if the map from $\mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ defined by $(x, y) \mapsto x^t Ay$ is positive definite).

(b) *Sylvester's Criterion* states that a symmetric matrix is positive definite if and only if all principle minors of A (that is, submatrices obtained by deleting the last k rows and the last k columns for some $0 \leq k < m$) have positive determinant. Prove this result when A is a 2×2 symmetric matrix.

Problem 3. Let $\phi : \mathbb{R}^m \to \mathbb{R}$ be function of class C^2. Let x be a critical point of ϕ (i.e. suppose $\nabla\phi(x) = 0$). Let

$$H = \left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}(x) \right)$$

be the Hessian of ϕ at x. In class we defined $\frac{\partial^2 \phi}{\partial v^2}(x)$ to be $\frac{d^2}{dt^2}[\phi(x + tv)]_{t=0}$. Prove that

$$\frac{\partial^2 \phi}{\partial v^2}(x) = v^t Hv$$

for any $v \in \mathbb{R}^m$. (Hint: it will be useful to prove that $\frac{\partial^2 \phi}{\partial v^2}(x) = \frac{\partial}{\partial v}(\frac{\partial}{\partial v}\phi)(x)$.)

Problem 4. Let $f^{-1}(x)$ be a regular level set in $\mathbb{R}^m$, and let x be a point in $f^{-1}(c)$. A *curve* is a map $\gamma : \mathbb{R} \to \mathbb{R}^m$ of class C^1. We say a curve γ is *based at* x if $\gamma(0) = x$. In this case we call $\gamma'(0) \in T_x\mathbb{R}^m$ the *initial velocity vector* of γ. We say a curve *lies in* $f^{-1}(c)$ if every point of its image is contained in $f^{-1}(c)$. Let V be the set of all initial velocity vectors of curves lying in $f^{-1}(c)$ and based at x. Prove that $V = T_x f^{-1}(c) := \ker Df(x)$. In particular:

(a) Show that V is a vector space. (One approach: First consider the set W, consisting of all velocity vectors lying in an open ball $U \subset \mathbb{R}^k$ based at a point $a \in U \subset \mathbb{R}^k$. Show that W is a vector space. Then use the parametrization of $f^{-1}(c)$ about x that we get from the implicit function theorem, in order to establish a bijection between elements of V and elements of W.)

(b) Show that $V \subset \ker Df(x)$.

(c) Show that $V \supset \ker Df(x)$.

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