PROBLEM SET 1 (VERSION 2)

INTRODUCTION TO MANIFOLDS

In class we stated the following two Taylor's theorems with remainder (each version contained an error, which two students thankfully spotted!). Update: some additional corrections have been marked in red.

Theorem 1 (Notationally taxing Taylor). *Let $U \subset \mathbb{R}^n$ be an open subset that is star-shaped with respect to a point $p = (p^1, \ldots, p^n) \in U$. Suppose $f : U \to \mathbb{R}$ is a C^∞ function on U. Let k be any positive integer. We then have*

$$f(x) = f(p) + \sum_{i=1}^{n}(x^i - p^i)\frac{\partial f}{\partial x^i}(p) + \ldots + \frac{1}{k!}\sum_{i_1,\ldots,i_k}(x^{i_1} - p^{i_1})\cdots(x^{i_k} - p^{i_k})\frac{\partial^k f}{\partial x^{i_1}\cdots\partial x^{i_k}}(p)$$
$$+\frac{1}{k!}\sum_{i_1,\ldots,i_{k+1}}(x^{i_1} - p^{i_1})\cdots(x^{i_{k+1}} - p^{i_{k+1}})\int_0^1(1-t)^k\frac{\partial^{k+1} f}{\partial x^{i_1}\cdots\partial x^{i_{k+1}}}(p + t(x-p))dt.$$

Theorem 2 (Bare bones Taylor). *Let $U \subset \mathbb{R}^n$ be an open subset that is star-shaped with respect to a point $p = (p^1, \ldots, p^n) \in U$. Suppose $f : U \to \mathbb{R}$ is a C^∞ function on U. Then there exist functions $g_1, \ldots, g_n$ in $C^\infty(U)$ such that*

$$f(x) = f(p) + \sum_{i=1}^{n}(x^i - p^i)g_i(x), \text{ with } g_i(p) = \frac{\partial f}{\partial x^i}(p).$$

Problem 1. Prove Theorem 1 for $k = 2$. It may help to define the path $\gamma(t) := p + t(x-p)$ with $0 \le t \le 1$, and to consider $\frac{d}{dt}f(\gamma(t))$, as we did in class.

Problem 2. Use Theorem 2 to construct functions $g_{i,j} \in C^\infty(U)$ for $1 \le i \le j \le n$ such that

$$f(x) = f(p) + \sum_{i=1}^{n}(x^i - p^i)\frac{\partial f}{\partial x^i}(p) + \sum_{i\le j}(x^i - p^i)(x^j - p^j)g_{i,j}(x), \text{ where}$$
$$g_{i,j}(p) = \begin{cases} \frac{\partial^2 f}{\partial x^i \partial x^j}(p) & \text{for } i < j, \text{ and} \\ \frac{1}{2}\frac{\partial^2 f}{\partial x^i \partial x^j}(p) & \text{for } i = j. \end{cases}$$

Problem 3. Prove that the following function is C^∞ at 0 but not real-analytic at 0:

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Problem 4. Prove that the set of derivations at a point $p \in \mathbb{R}^n$ forms a vector space.

Problem 5. Let A be an algebra over a field K. If D_1 and D_2 are derivations of A, show that $D_1 \circ D_2$ is not necessarily a derivation of A, but that $D_1 \circ D_2 - D_2 \circ D_1$ is always a derivation of A.

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