PROBLEM SET 12

INTRODUCTION TO MANIFOLDS

Suppose M is a connected non-orientable smooth manifold with atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha = (x^1_\alpha, \dots, x^n_\alpha))\}$. Consider the set

$$\widetilde{M} := \bigsqcup_{p \in M} \{\text{orientations of } T_pM\},$$

and let $\pi : \widetilde{M} \to M$ be the projection $[\omega_p] \mapsto p$. Here are four problems to help you understand the orientation covering $\widetilde{M}$.

Problem 1. For every α, let

$$U^+_\alpha = \{\phi^*_\alpha [dx^1_\alpha \wedge \ldots \wedge dx^n]_p : p \in U_\alpha\}$$

and let

$$U^-_\alpha = \{\phi^*_\alpha [-dx^1_\alpha \wedge \ldots \wedge dx^n]_p : p \in U_\alpha\}.$$

For any α, let $\bar{\phi}_\alpha := (-x^1_\alpha, x^2_\alpha, \dots, x^n_\alpha)$ be the opposite orientation chart to ϕ_α. Show that the collection

$$\widetilde{\mathcal{A}} := \{(U^+_\alpha, \phi^+_\alpha := \phi_\alpha \circ \pi)\} \cup \{(U^-_\alpha, \phi^-_\alpha := \bar{\phi}_\alpha \circ \pi)\}$$

gives $\widetilde{M}$ the structure of a smooth manifold.

Problem 2. Show that $\widetilde{A}$ is an *oriented* atlas on $\widetilde{M}$ (i.e., whenever two sets $U^\pm_\alpha$ and $U^\pm_\beta$ intersect, the transition map $\phi^\pm_\beta \circ (\phi^\pm_\alpha)^{-1}$ is orientation preserving).

Problem 3. Show that the manifold $\widetilde{M}$ is connected.

Problem 4. Describe the oriented atlas $\widetilde{\mathcal{A}}$ on $\widetilde{M}$ in the special case where M is a Möbius strip and $\mathcal{A}$ consists of two charts on M.

Now let $\alpha : \widetilde{M} \to \widetilde{M}$ be the map that interchanges the two points in each fiber of π. The following two problems will determine the top cohomology of the non-orientable manifold M.

Problem 5. Prove that α is an orientation-reversing diffeomorphism of $\widetilde{M}$.

Problem 6. Now suppose M has dimension n. Since $\widetilde{M}$ is orientable, there exists an isomorphism from $H^n(\widetilde{M})$ to $\mathbb{R}$ defined by

$$[\omega] \mapsto \int_{\widetilde{M}} \omega,$$

by a result mentioned in class. Assuming this isomorphism, prove that $H^n(M) = 0$.

The following three problems relate to integral curves and flows, and are taken from Chapter 12 of [Lee].

Problem 7. Recall that a vector field is *complete* if it generates a global flow; i.e. if all integral curves are defined for all $t \in \mathbb{R}$. Show that every smooth vector field with compact support is complete.

Problem 8. Let M be a connected smooth manifold. Show that the group of diffeomorphisms of M acts transitively on M. (For hints see Problem 12.3 in [Lee].)

Problem 9. Let θ be a flow on an oriented manifold. Show that for each $t \in \mathbb{R}$, θ_t is orientation-preserving wherever it is defined.

Date: November 16, 2015.