PROBLEM SET 2

INTRODUCTION TO MANIFOLDS

Problem 1. (a) Suppose $\beta \in L_k(\mathbb{R}^n)$ is symmetric and $\gamma \in L_k(\mathbb{R}^n)$ is alternating. Compute $S\beta, A\beta, S\omega$, and $A\omega$.

(b) Show that $\alpha^1 \otimes \alpha^2 \otimes \alpha^3$ cannot be expressed as the sum of a symmetric part and an alternating part, where $\{\alpha^i\}$ is the standard basis for $(\mathbb{R}^n)^*$.

Problem 2. (a) Prove that $\beta \wedge \beta = 0$ for all $\beta \in A_k(\mathbb{R}^n)$ if k is odd.

(b) Produce an element $\beta \in A_2(\mathbb{R}^n)$ such that $\beta \wedge \beta \neq 0$.

Problem 3. Suppose $\beta^1, \ldots, \beta^k$ and $\gamma^1, \ldots, \gamma^k$ are elements of $L_1(\mathbb{R}^n) = (\mathbb{R}^n)^*$.

(a) Prove that if

$$\beta^i = \sum_{j=1}^{k} a^i_j \gamma^j, \text{ for } i = 1, \ldots, k,$$

for some $k \times k$ matrix $A = [a^i_j]$, then

$$\beta^1 \wedge \ldots \wedge \beta^k = \det(A)\gamma^1 \wedge \ldots \wedge \gamma^k.$$

(b) Prove that $\beta^1 \wedge \ldots \wedge \beta^k \neq 0$ if and only if $\beta^1, \ldots, \beta^k$ are linearly independent in $(\mathbb{R}^n)^*$.

(c) Assuming $\beta^1, \ldots, \beta^k$ and $\gamma^1, \ldots, \gamma^k$ are indeed linearly independent, show that they have the same span in $(\mathbb{R}^n)^*$ if and only if

$$\beta^1 \wedge \ldots \wedge \beta^k = C\gamma^1 \wedge \ldots \wedge \gamma^k$$

for some nonzero number $C \in \mathbb{R}$.

Problem 4. Let $\{\alpha^1, \alpha^2, \alpha^3\}$ be the dual basis to the standard basis $\{e_1, e_2, e_3\}$ on $\mathbb{R}^3$. To each 1-covector $\alpha = a_1\alpha^1 + a_2\alpha^2 + a_3\alpha^3 \in A_1(\mathbb{R}^3)$, we associate a vector $\mathbf{v}_\alpha := a_1e_1 + a_2e_2 + a_3e_3 \in \mathbb{R}^3$. To each 2-covector $\gamma = c_1\alpha^2 \wedge \alpha^3 + c_2\alpha^3 \wedge \alpha^1 + c_3\alpha^1 \wedge \alpha^2 \in A_2(\mathbb{R}^3)$, we associate a vector $\mathbf{v}_\gamma := c_1e_1 + c_2e_2 + c_3e_3 \in \mathbb{R}^3$. Show that for any 1-covectors α and β, we then have $\mathbf{v}_{\alpha\wedge\beta} = \mathbf{v}_\alpha \times \mathbf{v}_\beta$.

The following two problems are optional. They will not contribute to or detract from your grade, but you are encouraged to attempt them.

Challenge 1. Prove Poincare's Lemma for 1-forms on $\mathbb{R}^3$: If ω is a differential 1-form on $\mathbb{R}^3$ such that $d\omega = 0$, then $\omega = df$ for some $f \in C^\infty(\mathbb{R}^3)$.

Challenge 2. Recall that $\omega \in A_k(V)$ is decomposable if $\omega = \beta^1 \wedge \ldots \wedge \beta^k$ for some 1-covectors $\beta^1, \ldots, \beta^k$ on V. Prove that $\omega \in A_{n-1}(\mathbb{R}^n)$ is always decomposable.

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