PROBLEM SET 6

INTRODUCTION TO MANIFOLDS

Note that this problem set will not be collected. Nevertheless, it is recommended that you work out all of these problems.

Problem 1. Prove that if N is a compact manifold, then a one-to-one immersion $f : N \to M$ is an embedding.

Problem 2. Let $(U, \phi) = (U, x^1, \ldots, x^n)$ and $(V, \psi) = (V, y^1, \ldots, y^n)$ be overlapping coordinate charts on a manifold N. They induce coordinate charts $(TU, \tilde{\phi})$ and $(TV, \tilde{\psi})$ on the tangent bundle TN, where $\tilde{\phi} = (x^1, \ldots, x^n, a^1, \ldots, a^n)$ and $\tilde{\psi} = (y^1, \ldots, y^n, b^1, \ldots, b^n)$.

(1) Compute the Jacobian matrix $J(\tilde{\psi} \circ \tilde{\phi}^{-1})|_{\tilde{\phi}(v_p)}$ of the transition function $\tilde{\psi} \circ \tilde{\phi}^{-1}$ at a point $\tilde{\phi}(v_p)$, where v_p is some point in TN.

(2) Show that
$$\det J(\tilde{\psi} \circ \tilde{\phi}^{-1})|_{\tilde{\phi}(v_p)} = \left(\det J(\psi \circ \phi^{-1})|_{\phi(p)}\right)^2 .$$

Problem 3. (a) Let X be the vector field $x \frac{d}{dx}$ on $\mathbb{R}$. For each $p \in \mathbb{R}$, find the maximal integral curve $c(t)$ of X with $c(0) = p$.

(b) Let X be the vector field $x^2 \frac{d}{dx}$ on $\mathbb{R}$. For each $p \in \mathbb{R}_{>0}$, find the maximal integral curve $c(t)$ of X with $c(0) = p$.

Problem 4. Consider two smooth vector fields X, Y on $\mathbb{R}^n$:
$$X = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}, \qquad Y = \sum_{i=1}^{n} b^i \frac{\partial}{\partial x^i},$$
where a^i, b^i are smooth functions on $\mathbb{R}^n$. Since $[X, Y]$ is also a smooth vector field on $\mathbb{R}^n$,
$$[X, Y] = \sum_{k=1}^{n} c^k \frac{\partial}{\partial x^k}$$
for some smooth functions c^k on $\mathbb{R}^n$. Find a formula for c^k in terms of a^i and b^i.

Problem 5. Let $F : N \to M$ be a smooth diffeomorphism manifolds. Prove that if X and Y are smooth vector fields on N, then
$$F_*[X, Y] = [F_*X, F_*Y].$$

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