THE ORIENTATION COVERING

INTRODUCTION TO MANIFOLDS

Suppose M is a connected non-orientable smooth manifold with atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha = (x_\alpha^1, \ldots, x_\alpha^n))\}$. Consider the set

$$\widetilde{M} := \bigsqcup_{p \in M} \{\text{orientations of } T_pM\},$$

and let $\pi : \widetilde{M} \to M$ be the projection $[\omega_p] \mapsto p$. Here are some problems to help you understand the orientation covering $\widetilde{M}$.

Problem 1. For every α, let

$$U_\alpha^+ = \{\phi_\alpha^*[dx_\alpha^1 \wedge \ldots \wedge dx^n]_p : p \in U_\alpha\}$$

and let

$$U_\alpha^- = \{\phi_\alpha^*[-dx_\alpha^1 \wedge \ldots \wedge dx^n]_p : p \in U_\alpha\}.$$

For any α, let $\bar{\phi}_\alpha := (-x_\alpha^1, x_\alpha^2, \ldots, x_\alpha^n)$ be the opposite orientation chart to ϕ_α. Show that the collection

$$\widetilde{\mathcal{A}} := \{(U_\alpha^+, \phi_\alpha^+ := \phi_\alpha \circ \pi)\} \cup \{(U_\alpha^-, \phi_\alpha^- := \bar{\phi}_\alpha \circ \pi)\}$$

gives $\widetilde{M}$ the structure of a smooth manifold.

Problem 2. Show that $\widetilde{A}$ is an *oriented* atlas on $\widetilde{M}$ (i.e., whenever two sets $U_\alpha^\pm$ and $U_\beta^\pm$ intersect, the transition map $\phi_\beta^\pm \circ (\phi_\alpha^\pm)^{-1}$ is orientation preserving).

Problem 3. Show that the manifold $\widetilde{M}$ is connected.

Problem 4. Describe the oriented atlas $\widetilde{\mathcal{A}}$ on $\widetilde{M}$ in the special case where M is a Möbius strip and $\mathcal{A}$ consists of two charts on M.

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