PROBLEM SET 1

INTRODUCTION TO MANIFOLDS

Theorem 1 (Taylor's Theorem with Remainder)**.** *Let $U \subset \mathbb{R}^n$ be an open subset that is star-shaped with respect to a point $p = (p^1, \ldots, p^n) \in U$. Suppose $f : U \to \mathbb{R}$ is a C^∞ function on U. Let k be any positive integer. We then have*

$$f(x) = f(p) + \sum_{i=1}^{n}(x^i - p^i)\frac{\partial f}{\partial x^i}(p) + \ldots + \frac{1}{k!}\sum_{i_1,\ldots,i_k}(x^{i_1} - p^{i_1})\cdots(x^{i_k} - p^{i_k})\frac{\partial^k f}{\partial x^{i_1}\cdots\partial x^{i_k}}(p)$$

$$+\frac{1}{k!}\sum_{i_1,\ldots,i_{k+1}}(x^{i_1} - p^{i_1})\cdots(x^{i_{k+1}} - p^{i_{k+1}})\int_0^1(1-t)^k\frac{\partial^{k+1} f}{\partial x^{i_1}\cdots\partial x^{i_{k+1}}}(p + t(x-p))dt.$$

Problem 1. Prove Theorem 1 for $k = 2$. It may help to define the path $\gamma(t) := p + t(x - p)$ with $0 \le t \le 1$, and to consider $\frac{d}{dt}f(\gamma(t))$, as in the proof of the other formulation of Taylor's theorem in the textbook.

Problem 2. Prove that the following function is C^∞ at 0 but not real-analytic at 0:

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Problem 3. Prove that the set of all point derivations of $C_p^\infty(\mathbb{R}^n)$ forms a vector space.

Problem 4. Let A be an algebra over a field K. If D_1 and D_2 are derivations of A, show that $D_1 \circ D_2$ is not necessarily a derivation of A, but that $D_1 \circ D_2 - D_2 \circ D_1$ is always a derivation of A.

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