# PROBLEM SET 4

INTRODUCTION TO MANIFOLDS

**Problem 1.** Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. For any $p \in \mathbb{R}^n$ there is a canonical identification $T_p(\mathbb{R}^n) \to \mathbb{R}^n$ given by

$$\sum a^i \frac{\partial}{\partial x^i}\bigg|_p \mapsto \langle a^1, \dots a^n \rangle.$$

Show that the differential $L_{*,p} : T_p(\mathbb{R}^n) \to T_{L(p)}(\mathbb{R}^m)$ is exactly the map $L$ under the above identification.

**Problem 2.** Given smooth manifolds $M$ and $N$, let $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ be the two projections. Prove that for any $(p, q) \in M \times N$,

$$\pi_{1*} \oplus \pi_{2*} : T_{(p,q)}(M \times N) \to T_p M \oplus T_q N$$

is an isomorphism.

**Problem 3.** Let $G$ be a Lie group with multiplication map $\mu : G \times G \to G$, inverse map $\iota : G \to G$, and identity $e$.

(a) Show that $\mu_{*,(e,e)}(X, Y) = X + Y$ for any $(X, Y) \in T_e G \oplus T_e G \cong T_{(e,e)}(G \times G)$.

(b) Show that $\iota_{*,e}(X) = -X$ for any $X \in T_e G$.

**Problem 4.** Let $M$ be a smooth manifold. A function $f : M \to \mathbb{R}$ is said to be a *local maximum* if there is a neighborhood $U$ of $p$ such that $f(p) \geq f(q)$ for all $q \in U$.

(a) Let $I$ be an open interval in $\mathbb{R}$. Prove that if a differentiable function $f : I \to \mathbb{R}$ has a local maximum at $p \in I$, then $f'(p) = 0$.

(b) Using (a), prove that a local maximum of a smooth function $f : M \to \mathbb{R}$ is a critical point of $f$.

**Problem 5.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. Prove that its graph $\Gamma(f) := \{(x, y, f(x, y)) \in \mathbb{R}^3\}$ is a regular submanifold of $\mathbb{R}^3$.

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