# PROBLEM SET 8

INTRODUCTION TO MANIFOLDS

**Problem 1** (Problem (7-9) of Lee)**.** Let $G$ be a Lie group, and let $G_0$ denote the connected component of the identity.

(a) Show that $G_0$ is an embedded Lie subgroup of $G$, and that each connected component of $G$ is diffeomorphic to $G_0$.

(b) If $H$ is any connected open subgroup of $G$, show that $H = G_0$.

**Problem 2** (Problem (7-13) of Lee)**.** Prove that $SO(3, \mathbb{R})$ is Lie isomorphic to $SU(2, \mathbb{C})/\{\pm I\}$ and diffeomorphic to $\mathbb{R}P^3$ (you can use the strategy outlined in Lee).

**Problem 3.** Let $G$ be a Lie group of dimension $n$ with Lie algebra $\mathfrak{g}$. For each $g \in G$, let $c_g := l_g \circ r_{g^{-1}} : G \to G$ be the corresponding conjugation map. Note that the differential at the identity $c_{g*} : \mathfrak{g} \to \mathfrak{g}$ is a linear isomorphism, and hence $c_{g*} \in GL(\mathfrak{g})$. Show that $Ad : G \to GL(\mathfrak{g})$ defined by $Ad(g) = c_{g*}$ is a homomorphism of Lie groups; i.e. that it is an abstract group homomorphism *and* a smooth map between manifolds.

**Problem 4.** (a) Let $S$ be a regular submanifold of codimension $k$ in a smooth manifold $M$ of dimension $n$. Let $(U, \phi) = (U, f^1, \ldots, f^n)$ be local coordinates about a point $p \in S$ such that $S \cap U$ is defined by the vanishing of $f^1, \ldots, f^k$. Show that $T_pS = \bigcap_{i=1}^k \ker(df^i)$.

(b) Note that a vector field $X$ on $M$ restricts to a vector field on a submanifold $S \subset M$ if $X_p \in T_pS$ for any $p \in S$. Construct a vector field on $\mathbb{R}^{2n}$ that restricts to a nowhere-vanishing vector field on the unit sphere $S^{2n-1} \subset \mathbb{R}^{2n}$ (you can use part (a) to show that it does indeed restrict to $S^{2n-1}$).

**Problem 5.** Using the fact that the pullback of a smooth $k$-form is a smooth $k$-form, prove that if $\pi : \tilde{M} \to M$ is a surjective submersion, then the pullback map $\pi^* : \Omega^*(M) \to \Omega^*(\tilde{M})$ is an injective algebra homomorphism.

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